Spring 2011 $\boxed{\#1}$

$$\omega = x_1\, dx_2 \wedge dx_3 \wedge dx_4 + x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4$$

Compute $\int_{S^3} \omega$.

$$d\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4$$

$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

By stokes,

$$\int_{S^3} \omega = \int_{\partial B^4} \omega = \int_{B^4} d\omega = \int_{B^4} dx_1 dx_2 dx_3 dx_4 = \boxed{\mathrm{vol}(B^4)}.$$

Spring 2011 $\boxed{\#2}$

$M = \{(x,y) : x, y \in \mathbb{R}^3,\ \|x\| = 1,\ \|y\| = 1,\ \langle x,y\rangle = 0\}$.

Claim: M smooth cpct embedded submfd of $\mathbb{R}^6$ and can be identified with unit tangent bundle of S^2.

Define $M_1 = \{(x,y) : x,y \in \mathbb{R}^3,\ \|x\| = 1\}$ and $f_1 : \mathbb{R}^6 \longrightarrow \mathbb{R}$ by $f_1(x,y) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2$.

Then $M_1 = f_1^{-1}(1)$ and $(df_1)_p = [2x_1 \ 2x_2 \ 2x_3 \ 0 \ 0 \ 0] \neq 0$ $\forall p \in M_1$. Thus M_1 submfd of $\mathbb{R}^6$.

Define $M_2 = \{(x,y) \in M_1 : \|y\| = 1\}$ and $f_2 : M_1 \to \mathbb{R}$ by $f_2(x,y) = \|y\|^2 = y_1^2 + y_2^2 + y_3^2$. Then $M_2 = f_2^{-1}(1)$ and $(df_2)_p = [0 \ 0 \ 0 \ 2y_1 \ 2y_2 \ 2y_3] \neq 0$ $\forall p \in M_2$.

Thus, M_2 submfd of M_1.

Finally define $f : M_2 \longrightarrow \mathbb{R}$ by $f(x,y) = \langle x,y\rangle$

Then $f^{-1}(0) = M$ and $df_p = [y_1 \ y_2 \ y_3 \ x_1 \ x_2 \ x_3] \neq 0$ $\forall p \in M \Rightarrow M$ submfd of M_2 and therefore of $\mathbb{R}^6$.

And it must be cpct since $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2 = 2$.

We can identify M with the unit tangent bundle by identifying x with a point on the sphere S^2 and y with a unit tangent vector at x, since the normal vector at x is just x and all tangent vects char. by $\langle x,y\rangle = 0$.

Spring 2011 $\boxed{\#3}$

$\mathbb{RP}^n = S^n/\sim$, $x \sim -x$.

(a) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$

(b) CW decomp. of $\mathbb{RP}^n$ $n \geq 1$

(c) Use CW decomp to compute $H_k(\mathbb{RP}^n)$, $k \geq 0$.

(d) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

(a) $\mathbb{RP}^1 \cong S^1$ so $\pi_1(\mathbb{RP}^1) \approx \pi_1(S^1) = \mathbb{Z}$.

For $n > 1$, S^n is a 2-sheeted cover of $\mathbb{RP}^n$ and S^n is simply connected $\Rightarrow$ it is a universal cover.

Thus the trivial subgroup has index 2 in $\pi_1(\mathbb{RP}^n)$.

That means $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

In sum, $\boxed{\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & n=1 \\ \mathbb{Z}_2 & n>1 \end{cases}}$

(b) For $n \geq 2$, $\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup D^n$ where $\partial D^n = S^{n-1}$ is glued via the quotient map $q: S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ identifying antipodal points. $\mathbb{RP}^1$ is composed of a 0-cell and a 1-cell, so $\mathbb{RP}^n$ is composed of an i-cell for each $i \in \{0, \dots, n\}$, glued as above.

(c) $H_k(\mathbb{RP}^n) = 0$ for $k > n$. For $k \leq n$ we have

$H_k(\mathbb{RP}^n) = H_k(X^k) = H_k(\mathbb{RP}^k)$ where X^k is the k-dim'l subcomplex of $\mathbb{RP}^n$.

$H_1(\mathbb{RP}^1) \approx H_1(S^1) = \mathbb{Z}$.

For $K > 1$, we have relative homology LES:

$$\cdots \longrightarrow H_K(X^{K-1}) \longrightarrow H_K(\mathbb{RP}^K) \longrightarrow H_K(\mathbb{RP}^K, X^{K-1}) \longrightarrow H_{K-1}(X^{K-1}) \longrightarrow \cdots$$

for X^{K-1} the $(K-1)$-dim'l subcomplex of $\mathbb{RP}^K$.

$$H_K(\mathbb{RP}^K, X^{K-1}) \approx \tilde{H}_K(\mathbb{RP}^K / X^{K-1}) \approx \tilde{H}_K(S^K) = \mathbb{Z}$$

And $H_{K-1}(\mathbb{RP}^K) \approx H_{K-1}(\mathbb{RP}^{K-1})$, so we get

$$0 \longrightarrow H_K(\mathbb{RP}^K) \longrightarrow \mathbb{Z} \longrightarrow H_{K-1}(\mathbb{RP}^{K-1}) \longrightarrow H_{K-1}(\mathbb{RP}^K)$$

The map $H_{K-1}(\mathbb{RP}^{K-1}) \xrightarrow{i_*} H_{K-1}(\mathbb{RP}^K)$ is the 0-map b/c a $(K-1)$ chain in $\mathbb{RP}^{K-1}$ included in $\mathbb{RP}^K$ is the boundary of a K-dim'l cell in the CW decomp.

Thus, $\quad H_K(\mathbb{RP}^K) \hookrightarrow \mathbb{Z} \twoheadrightarrow H_{K-1}(\mathbb{RP}^{K-1})$.

Hence, $H_K(\mathbb{RP}^K) = \begin{cases} \mathbb{Z} & \text{if } H_{K-1}(\mathbb{RP}^{K-1}) = 0 \\ 0 & \text{if } H_{K-1}(\mathbb{RP}^{K-1}) = \mathbb{Z} \end{cases}$

In sum, $H_K(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & K = 0 \\ 0 & K > n \\ \mathbb{Z} & K \text{ odd}, \ 0 < K \leq n \\ 0 & K \text{ even}, \ 0 < K \leq n \end{cases}$

for $n \geq 1$

(d) Hence, since $H_n(M^n) = \mathbb{Z} \iff M^n$ orientable, for M^n an n dim'l closed, connected mfd, then $\mathbb{RP}^n$ orientable when n odd.

Spring 2011 $\boxed{\#4}$

$f: X \to Y$ continuous. $C_f = \big((X \times [0,1]) \sqcup Y\big)/\sim$

$(x,1) \sim f(x)$ $\forall x \in X$. $(x,0) \sim (x',0)$ $\forall x, x' \in X$.

Claim: $\exists$ LES:

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots$$

Let $A = X \times [0, 3/4]/\sim \;\simeq \{*\}$ $\qquad H_i(A) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$

$B = X \times [1/4, 1]/\sim \;\simeq Y$

Then $A \cap B \simeq X$, and by M.V. we get L.E.S.

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(C_f) \to \cdots$$

$\quad\;\wr\wr$

$$\cdots \to H_i(X) \to H_i(\{*\}) \oplus H_i(Y) \to H_i(C_f) \to \cdots$$

For $i > 0$ this is

$$\cdots \to H_i(X) \to H_i(Y) \to H_i(C_f) \to \cdots$$

and for $i = 0$ this is

$$\cdots \to H_0(X) \to \mathbb{Z} \oplus H_0(Y) \to H_0(C_f) \to 0$$

Thus, we can switch out this section for

$$\cdots \to H_0(X) \to H_0(Y) \to \tilde{H}_0(C_f) \to 0$$

and the whole LES is

$$\cdots \longrightarrow H_i(X) \longrightarrow H_i(Y) \longrightarrow \tilde{H}_i(C_f) \longrightarrow \cdots$$

since $\tilde{H}_i(C_f) = H_i(C_f)$ for $i > 0$.

We can make the switch because

$H_0(X) \longrightarrow \mathbb{Z} \oplus H_0(Y)$ and $H_0(X) \longrightarrow H_0(Y)$ both have kernel 0. and $\operatorname{im}(\mathbb{Z} \oplus H_0(Y) \longrightarrow H_0(C_f))$

$= H_0(C_f) \Rightarrow \operatorname{im}(H_0(Y) \longrightarrow H_0(C_f)) = \tilde{H}_0(C_f)$.

The last thing we check is that

$H_i(X) \longrightarrow H_i(Y)$ is the map f_*

We have:

$$\begin{tikzcd}
H_i(A \cap B) \arrow[r, "{j_*, j'_*}"] & H_i(A) \oplus H_i(B) \arrow[d, "r_*"] \\
H_i(X) \arrow[u, "i_*"] \arrow[r, dashed] & H_i(Y)
\end{tikzcd}$$

where i, j, j' are inclusions and r is a retraction. from $B \simeq Y$.

Then $(r \circ j' \circ i)x = r(x) = f(x)$ since the retraction just follows f into Y. Hence the map is indeed f_*.

Spring 2011 $\boxed{\#5}$

Recall a Lie Group G is a smooth mfd which is also a group, and whose group operations multiplication and inverse are smooth maps.

Claim: G connected lie group $\Rightarrow \pi_1(G)$ abelian.

We can choose the basepoint to be the identity element in G, e. Let $[f], [g] \in \pi_1(G, e)$.

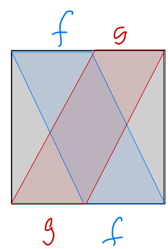
Then $f, g : I \longrightarrow G$ with $f(0) = f(1) = g(0) = g(1) = e$.

Define $H : I \times I \longrightarrow G$ by $H(s,t) = f_t(s)\, g_t(s)$

where $f_t(s) = \begin{cases} Ce(s) & s \in [0, t/2] \\ f(2s-t) & s \in [t/2, (t+1)/2] \\ Ce(s) & s \in [(t+1)/2, 1] \end{cases}$ $\qquad f\left(2\left(\frac{t}{2}\right) - t\right) = f(0) = e$
$\qquad f\left(2\left(\frac{t+1}{2}\right) - t\right) = f(1) = e$

and $g_t(s) = \begin{cases} Ce(s) & s \in [0, (1-t)/2] \\ g(2s+t-1) & s \in [(1-t)/2, (2-t)/2] \\ Ce(s) & s \in [(2-t)/2, 1] \end{cases}$ $\qquad g\left(2\left(\frac{1-t}{2}\right) + t - 1\right) = g(0) = e$
$\qquad g\left(2\left(\frac{2-t}{2}\right) + t - 1\right) = g(1) = e$

$H(s,0) = f_0(s)\, g_0(s) = \left\{ \begin{matrix} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{matrix} \right\} = (f \cdot g)(s)$



$H(s,1) = f_1(s)\, g_1(s) = \left\{ \begin{matrix} f(2s-1) & s \in [1/2, 1] \\ g(2s) & s \in [0, 1/2] \end{matrix} \right\} = (g \cdot f)(s)$

$H(0,t) = H(1,t) = e$. Hence $f \cdot g \simeq g \cdot f$ and $\pi_1(G)$ abelian.

Spring 2011 $\boxed{\#6}$

$M \subset \mathbb{R}^3$ embedded cpct oriented surface w/out bdry of genus $g \geq 1$. Show Gaussian curvature K of M must vanish somewhere.

Gauss Bonet gives

$$\int_M K \, dA = 2\pi \chi(M) = 2\pi(2-2g) \leq 0$$

So K must be negative somewhere on M.

All mfds with genus $g \geq 1$ also must have K positive somewhere on M. Thus it must vanish somewhere by continuity.