Geometry/Topology Qualifying Exam

Spring 2011

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Let $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ be the 3-dimensional sphere, oriented as the boundary of the unit ball B^4 in $\mathbb{R}^4$ with the standard orientation. Compute $\int_{S^3} \omega$, where

$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$

(You may leave your answer in terms of volumes $vol(S^n)$ and $vol(B^n)$.)

2. (10 pts) Let $M = \{(x, y) \mid x, y \in \mathbb{R}^3, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = 0\}$, where $\langle x, y\rangle$ is the standard inner product on $\mathbb{R}^3$. Show that M is a smooth compact embedded submanifold of $\mathbb{R}^6$ and explain how M can be identified with the unit tangent bundle of S^2.

check

3. (20 pts) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.
   (a) (5 pts) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$.
   (b) (5 pts) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.
   (c) (5 pts) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.
   (d) (5 pts) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

4. (10 pts) Given a continuous map $f : X \to Y$ between topological spaces, define

$$C_f = \left((X \times [0,1]) \coprod Y\right)/\sim,$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Then prove that there is a long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$

where f_* is the map on homology induced from f and $\tilde{H}_i$ denotes the ith reduced homology group.

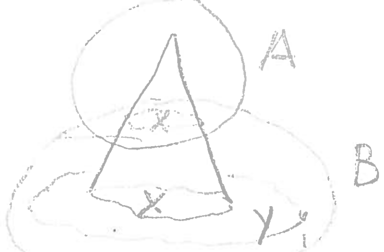
5. (10 pts) Prove that the fundamental group of a connected Lie group G is abelian. (A *Lie group* G is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between fg and gf, where f and g are loops in G.]

6. (10 pts) Let $M \subset \mathbb{R}^3$ be an embedded compact oriented surface (without boundary) of genus $g \geq 1$. Show that the Gaussian curvature κ of M must vanish somewhere on M.

→ means we have pt. of positive curvature

($g > 0$, cpt, orntble)

4 =



→ $A \cap B \simeq X$

$A \simeq pt$

$B \simeq Y$

$A \cup B \simeq C_f$

1. Stokes
2. Regular Value Theorem
3. Cellular Homology
4. Snake Lemma (short exact seq of chain complex $\rightsquigarrow$ long exact seq of homology ; Hatcher p. 116–117)
5. Let basepoint be the identity. Use $I \times I \to G$ trick
6. Gauss-Bonnett, must have a point of positive curvature.

① $S^3 \subseteq \mathbb{R}^4$ oriented as ∂B^4. Compute $\int_{S^3}\omega$ where
$\omega = x_1\, dx_2 \wedge dx_3 \wedge dx_4 + x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4$

$$d\omega = d(x_1 dx_2 \wedge dx_3 \wedge dx_4) + d(x_2 dx_1 \wedge dx_3 \wedge dx_4) + d(x_3 dx_1 \wedge dx_2 \wedge dx_4)$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

Now apply Stokes:

$$\int_{S^3}\omega = \int_{\partial B^4}\omega = \int_{B^4} d\omega = \int_{B^4} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = \underline{\mathrm{Vol}(B^4)}$$

② $M = \{(x,y) : x, y \in \mathbb{R}^3,\ \|x\| = \|y\| = 1,\ \langle x,y\rangle = 0\}$
Show that M is a smooth, compact, embedded submanifold of $\mathbb{R}^6$ and show that we can identify $M \approx T^1S^2$, unit tan. bundle of S^2

Consider the map $f: S^2 \times S^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto \langle x,y\rangle$

Clearly we have $f^{-1}(0) = M$, hence M is closed subspace of compact space, hence compact.

∴ Now apply the Regular Value Theorem; if 0 is regular value then $f^{-1}(0)$ is embedded submfld. 0 is regular if T_pf surjective $\forall p \in f^{-1}(0)$
Consider f on $\mathbb{R}^3 \times \mathbb{R}^3$ and it is given by $f(x,y) = x_1y_1 + x_2y_2 + x_3y_3$.
hence $T_pf = [y_1\ y_2\ y_3\ x_1\ x_2\ x_3]$ where $p = (x_1, x_2, x_3, y_1, y_2, y_3)$.

Recall that $p \in f^{-1}(0) \subseteq S^2 \times S^2 \Rightarrow \exists i,j$ s.t. $x_i \neq 0,\ y_j \neq 0$,
hence $T_pf \not\equiv 0$, hence T_pf is surjective by linearity & codom $= \mathbb{R}$

→ we may make the identification $M \cong T^1S^2$ by looking at the defns:
$T^1S^2 = \{(x,v) : x \in S^2,\ v \in T_xS^2 \subseteq \mathbb{R}^3,\ \|v\| = 1\}$
$M = \{(x,y) : x,y \in \mathbb{R}^3,\ \|x\| = \|y\| = 1,\ \langle x,y\rangle = 0\}$

- $x \in S^2,\ v \in T_xS^2 \Leftrightarrow x \perp v \Leftrightarrow \langle x,v\rangle = 0$
- $\|v\| = 1 \Leftrightarrow v \in S^2$

③ $\mathbb{R}P^n = S^n/\sim$ where $x \sim -x$ (antipodes identified)

(a) Use covering spaces to compute $\pi_1(\mathbb{R}P^n)$

• Case $n \geq 2$: The antipodal map $\alpha(x) = -x$ gives an action of $\mathbb{Z}_2$ on S^n (since $\langle \alpha \rangle \cong \mathbb{Z}_2$), hence $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ the orbit space under this action.

→ Now, if each $x \in S^n$ has a nbhd $U \ni x$ st. $g(U)$ are disjoint for all $g \in \mathbb{Z}_2$, we get that:

"covering space action"

(1) $\pi: S^n \to S^n/\mathbb{Z}_2$ normal covering space

(2) $\mathbb{Z}_2$ is the group of deck transformations of π

(3) $\mathbb{Z}_2 \cong \pi_1(S^n/\mathbb{Z}_2) / \pi_*(\pi_1(S^n))$ (since S^n path-conn, loc path-conn.)

→ Each hemisphere of S^n is disjoint from its image under α, the generator for $\mathbb{Z}_2$ (i.e. id(hemisphere) $\cap$ α(hemisphere) $= \emptyset$), hence we get statement (3), i.e. $\mathbb{Z}_2 \cong \pi_1(S^n/\mathbb{Z}_2) / \pi_*(\pi_1(S^n)) \cong \pi_1(\mathbb{R}P^n) / \pi_*(1) \cong \pi_1(\mathbb{R}P^n)$

$\Rightarrow \underline{\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2}$.

• Case $n = 1$: In this case $\mathbb{R}P^1$ & S^1 are homeomorphic, hence $\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z} \Rightarrow \underline{\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}}$

ALT. • explicit version for $n \geq 2$:

Let g be a path in S^n connecting x and $\alpha(x)$; recall $\pi: S^n \to \mathbb{R}P^n$

then define $\gamma = \pi \circ g$, a loop in $\mathbb{R}P^n$ (since $\alpha(x) \sim x$ here) that generates $\pi_1(\mathbb{R}P^n)$

Now consider γ^2 and lift it to S^n:

$$\begin{array}{ccc} & & S^n \\ & \overset{\tilde{\gamma}^2}{\nearrow} & \downarrow \pi \\ [0,1] & \xrightarrow[\gamma^2]{} & \mathbb{R}P^n \end{array}$$

$\tilde{\gamma}^2$ is a loop in S^n since it goes from x to $\alpha(x)$ to $\alpha(\alpha(x)) = x$, hence it is homotopic to const. since $\pi_1(S^n) = 1$, hence

By the uniqueness of lifting, γ^2 must also be homotopic to const, hence $\gamma^2 \simeq$ const $\Rightarrow \gamma$ order 2 $\Rightarrow \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$

(b) Give cell decomposition of $\mathbb{R}P^n$ for $n \geq 1$.

$\mathbb{R}P^n$ has a cell structure with 1 cell of each dimension $\leq n$.
See that $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_p D^n$ with attaching map the two-sheeted covering $p: \partial D^n = S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$

(c) Use the cell decomposition to compute $H_k(\mathbb{R}P^n)$, $k \geq 0$

The cellular chain complex $C_k(\mathbb{R}P^n)$ has one cell e^k in each dim, $k \leq n$
hence $C_k(\mathbb{R}P^n) \cong \mathbb{Z}\{e_k\}$ for each $0 \leq k \leq n$
we know e^k has attaching map $p_k: \underset{\text{boundary of } e^k}{S^{k-1}} \longrightarrow \mathbb{R}P^{k-1}$ (2-sheeted cover)

→ apply Cellular Boundary Formula:

$$d_k(e^k) = \sum_\beta d_\beta e_\beta^{n-1} \text{ where } d_\beta = \deg\left(\partial(e^k) = S^{k-1} \xrightarrow[\text{attaching map of } e^k]{} \underset{(k-1)\text{skeleton}}{X^{k-1}} \xrightarrow[\text{quotient sending } X^{k-1} - e_\beta^{k-1} \text{ to pt}]{} S_\beta^{k-1} = e_\beta^{k-1}\right)$$

Recall that $\mathbb{R}P^n$ has k-skeleton the $\mathbb{R}P^k$, $k \leq n$.
So now to find $d_k(e^k) = d_\beta e^{k-1}$, we need to find d_β,
the degree of

$$S^{k-1} \xrightarrow[\text{attaching map of } e^k]{p_k} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \simeq S^{k-1}$$

q: quotient sending $X^{k-1} - e^{k-1} \cong \mathbb{R}P^{k-1} - e^{k-1} \cong \mathbb{R}P^{k-2}$ to point

Now let $N, S \subseteq S^{k-1}$ be the hemispheres of S^{k-1}.
Then $q \circ p_k|_N$ & $q \circ p_k|_S$ are homeomorphisms such that $q \circ p_k \circ \alpha|_N = q \circ p_k|_S$, ($\alpha$: antipode of S^{k-1})
hence $\deg(q \circ p_k) = \deg(q \circ p_k|_N) + \deg(q \circ p_k|_S) = \deg(id) + \deg(\alpha)$
$= (-1) + (-1)^k$

degree computed from local degree ~> Prop 2.30 Hatcher

Fact: $\deg \alpha = (-1)^k$ for S^{k-1}

$$\Rightarrow d_k(e^k) = \begin{cases} 2e^{k-1}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

So we have the chain complexes:

$$n \text{ even} \quad 0 \to \mathbb{Z} \xrightarrow{\cdot 2 = d_n} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} 0$$

$$n \text{ odd} \quad 0 \to \mathbb{Z} \xrightarrow{\cdot 0 = d_n} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} 0$$

$$\Rightarrow \boxed{H_k(\mathbb{RP}^n) = \ker d_k / \operatorname{im} d_{k+1} \cong \begin{cases} \mathbb{Z} & k=0 \text{ or } n \text{ odd} \\ \mathbb{Z}_2 & k \text{ odd},\ 0<k<n \\ 0 & \text{otherwise} \end{cases}}$$

(d) Prove $\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd.

First, note that $\alpha: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ (antipode) is given by $\alpha = \operatorname{diag}(-1)$, hence orientation preserving $\Leftrightarrow 0 < \det \alpha = (-1)^{n+1} \Leftrightarrow n$ odd

($\Rightarrow$) $\mathbb{RP}^n$ orientable: Recall the quotient map $\pi: S^n \to \mathbb{RP}^n$

The orientation of $\mathbb{RP}^n$ induces orientation on S^n via π ...

$(v_1, \dots, v_n) \subseteq T_pS^n$ positively oriented basis iff $(T_p\pi(v_1), \dots, T_p\pi(v_n)) \subseteq T_{\pi(p)}\mathbb{RP}^n$ positively oriented.

Suppose n even; recall that α orientation reversing in this case, but now see that

$(T_p\pi(v_1), \dots, T_p\pi(v_n))$ and $(T_p\pi(\alpha(v_1)), \dots, T_p\pi(\alpha(v_n)))$

have the same orientation since $\alpha = id$ on $\mathbb{RP}^n$.

Therefore $(\alpha(v_1), \dots, \alpha(v_n))$ & $(v_1, \dots, v_n)$ have same orientation, which is a contradiction since n is even; thus n must be odd.

($\Leftarrow$) n odd; now we want to put an orientation on $\mathbb{RP}^n$

Define $(w_1, \dots, w_n) \subseteq T_{\pi(p)}\mathbb{RP}^n$ to be positively-oriented iff $\exists$ positively-oriented basis $(v_1, \dots, v_n) \subseteq T_pS^n$ such that $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (w_1, \dots, w_n)$.

This is well-def'd: Let $(v_1, \dots, v_n)$ & $(v_1', \dots, v_n')$ be orient. bases for T_pS^n, $T_{-p}S^n$ (resp.) in the same orient. class. Now recall $\pi \circ \alpha = \pi$, hence:

$$T_p\pi(v_i) = T_p(\pi \circ \alpha)(v_i) = T_{\alpha(p)}\pi \circ T_p\alpha(v_i) = T_{-p}\pi \circ T_p\alpha(v_i).$$

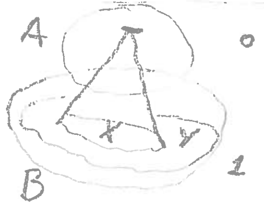
n odd $\Rightarrow \alpha$ orient. pres. $\Rightarrow T_p\alpha(v_i)$ & (v_i') same orient. class on $T_{-p}S^n$

$\Rightarrow T_p\pi(v_i)$ & $T_{-p}\pi(v_i')$ same orient. class on $T_{\pi(p)}\mathbb{RP}^n$.

4. $f: X \to Y$; $C_f = (X \times [0,1]) \sqcup Y / \sim$ where $(x,1) \sim f(x)$, $(x,0) \sim (x',0)$ $\forall x, x' \in X$

Show $\exists$ long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow[f_*]{} H_{i+1}(Y) \longrightarrow \tilde{H}_{i+1}(C_f) \longrightarrow H_i(X) \xrightarrow[f_*]{} H_i(Y) \longrightarrow \cdots$$

$C_f =$ 

Let $A = X \times [0, \frac{1}{2}+\varepsilon] / \sim$ Then $A \cup B = C_f$
$B = (X \times [\frac{1}{2}-\varepsilon, 1] \sqcup_f Y)$

Hence $A \cap B = X \times [-\varepsilon, \varepsilon] \simeq X$ and $A \simeq X \times \{0\} \simeq pt.$ by $\sim$.
Also, $B \simeq (X \times \{1\}) \sqcup Y \simeq (f(X)) \sqcup Y \simeq Y$ since, $f(X) \subseteq Y$.

Now apply Mayer-Vietoris:

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(A \cup B) \to H_{i-1}(A \cap B) \to \cdots$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(pt.) \oplus H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to H_0(pt.) \oplus H_0(Y) \to H_0(C_f) \to 0 \end{cases}$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to \mathbb{Z} \oplus H_0(Y) \to H_0(C_f) \to 0 \end{cases}$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(Y) \to \tilde{H}_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to H_0(Y) \to \tilde{H}_0(C_f) \to 0 \end{cases}$$

recall: $H_i(C_f) \cong \tilde{H}_i(C_f)$ $i>0$

$H_0(C_f) = \tilde{H}_0(C_f) \oplus \mathbb{Z}$.

and furthermore:

$$\cdots \to H_i(A \cup B) \xrightarrow{i^*} \overset{0}{\cancel{H_i(A)}} \oplus H_i(B) \to H_i(A \cup B) \to \cdots$$

where $i : A \cup B \hookrightarrow B$.

Now note that $A \cup B$ $\overset{\text{def.}}{\text{retracts}}$ to X and B $\overset{\text{def.}}{\text{retracts}}$ to Y.

So then:

$$\begin{tikzcd} A \cup B \arrow[r, hook] \arrow[d, "r_1"'] & B \arrow[d, "r_2"] \\ X \arrow[r] & Y \end{tikzcd}$$

will have commutative induced homology diagram since r_i are $\overset{\text{def.}}{\text{retractions}}$ (i.e. r_i^* are isomorphisms):

$$\begin{tikzcd} H_i(A \cup B) \arrow[r, "i^*"] \arrow[d, "r_1^*"', "\wr"] & H_i(B) \arrow[d, "\wr"', "r_2^*"] \\ H_i(X) \arrow[r, "f^*"'] & H_i(Y) \end{tikzcd}$$

Hence $(r_2^*)^{-1} \circ f \circ r_1^* = i^*$, hence these maps agree since r_i^* are isomorphisms and $Y \subseteq B$,

hence

$$\cdots \to H_i(X) \xrightarrow{f^*} H_i(Y) \to \tilde{H}_i(C_f) \to \cdots$$

as was desired.

(5) Prove that the fundamental group of a connected Lie group G is abelian :

Let f, g be loops in G based ~~at~~ at e, the identity elt. (WLOG)
We will show that $fg \simeq gf$ (homotopic) by writing an explicit homotopy.

Recall $f, g : I \to G$; now define

$$F : I \times I \to G, \quad (s,t) \mapsto f(s)g(t)$$

group multiplication is continuous, hence so is F.

See that
$$F(t,0) = F(t,1) = f(t)$$
$$F(1,t) = F(0,t) = g(t)$$

So we want to construct a homotopy from $F(t,0) \cdot F(1,t) = f \cdot g(t)$
to $F(0,t) \cdot F(t,1) = g \cdot f(t)$:

$$f \cdot g(t) = F(t,0) \cdot F(1,t) = \begin{cases} F(2t, 0), & 0 \le t \le \frac{1}{2} \\ F(1, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$g \cdot f(t) = F(0,t) \cdot F(t,1) = \begin{cases} F(0, 2t), & 0 \le t \le \frac{1}{2} \\ F(2t-1, 1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Then see that
$$G_s(t) = \begin{cases} F(2t(1-s), 2ts), & 0 \le t \le \frac{1}{2} \\ F((1-s) + (2t-1)s, (1-s)(2t-1) + s), & \frac{1}{2} \le t \le 1 \end{cases}$$

a homotopy between the two. - ?

or more simply: $G_s(t) = F((1-s)t, st) \cdot F((1-s)+st, s+(1-s)t)$,
which gives $G_0(t) = F(t,0) \cdot F(1,t) = f \cdot g$
$G_1(t) = F(0,t) \cdot F(t,1) = g \cdot f$

ok but this is loop composition

6.) $M \subseteq \mathbb{R}^3$ embedded, cpt, oriented surface, genus $g \geq 1$.
Show that Gaussian curvature K of M must be 0 somewhere on M:

Since M is cpt, orientable, positive genus, $\exists$ pt of positive curvature, i.e. $K > 0$ somewhere on M.
Now apply Gauss-Bonnet:

$$\int_M K\,dA = 2\pi \chi(M) = 2\pi(2-2g)$$
$$= 4\pi(1-g)$$
$$\leq 0$$

since $\chi(M) = 2-2g$ for genus g surface.

Hence K must be negative for some part of M, and then by the Intermediate Value Theorem, $\exists\, p \in M$ s.t. $K(p) = 0$.