

Geometry Spring 2011

① Let $S^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}$ oriented as the boundary of $B^4 = \{x \in \mathbb{R}^4 : \|x\| \leq 1\}$ w/ the standard orientation.

Compute $\int_{S^3} \omega$ where

$$\omega = X_1 dx_2 \wedge dx_3 \wedge dx_4 + X_2 dx_1 \wedge dx_3 \wedge dx_4 + X_3 dx_1 \wedge dx_2 \wedge dx_4$$

in terms of $\text{vol}(S^3)$ and $\text{vol}(B^4)$.

solution

$$\int_{S^3} \omega = \int_{B^4} d\omega$$

$$\begin{aligned} d\omega &= v - v + v \\ &= v. \end{aligned}$$

$v := dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$
standard volume form
on $\mathbb{R}^4$.

$$\text{vol}(B^4) := \int_{B^4} v$$

$$\int_{S^3} \omega = \text{vol}(B^4).$$



$$(2) \quad M = \{ (x, y) : x, y \in \mathbb{R}^3, \|x\|=1, \|y\|=1, \langle x, y \rangle = 0 \}$$

where $\langle x, y \rangle$ is the standard inner product in $\mathbb{R}^3$.

Show M is a smooth compact embedded submanifold of $\mathbb{R}^6$, and show M can be identified w/ the unit tangent bundle of S^2 .

solution $f: \mathbb{R}^6 \rightarrow \mathbb{R}^3$

$$f(x_1, x_2, x_3, y_1, y_2, y_3) = (x_1^2 + x_2^2 + x_3^2, y_1^2 + y_2^2 + y_3^2, x_1 y_1 + x_2 y_2 + x_3 y_3)$$

$$M = f^{-1}(\{(1, 1, 0)\}) \quad \text{Let } (x, y) \in M.$$

$$T_{(x, y)} f = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

Let $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3$
Let $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$

Want to solve $\begin{pmatrix} 2x_1 v_1 + 2x_2 v_2 + 2x_3 v_3 \\ 2y_1 v_1 + 2y_2 v_2 + 2y_3 v_3 \\ x_1 y_1 + x_2 y_2 + x_3 y_3 + v_1 x_1 + v_2 x_2 + v_3 x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Take $(v_1, v_2, v_3) = \frac{a}{2}(x_1, x_2, x_3)$ and

Since $|x|^2=1$ and $|y|^2=1$, the first and 2nd rows are l.i. Since $\langle x, y \rangle = 0$, the third row is l.i. from the first two rows. Therefore, the matrix has

rank 3 and M is an embedded submanifold of $\mathbb{R}^6$ of dimension 3. M is closed and contained in the 5-sphere of radius

$\sqrt{2}$ centered at the origin in $\mathbb{R}^6$; so M is bounded and closed, hence compact.

The unit tangent bundle of S^2

$$\text{is } T_1 S^2 = \{ (x, y) : x \in S^2, y \in T_x S^2 \text{ and } \|y\| = 1 \}$$

where we have identified $T_x S^2 \cong \ker T_x g \subset \mathbb{R}^3$ as the kernel of the derivative $T_x g : \mathbb{R}^3 \rightarrow \mathbb{R}$

of the function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $g(x) = \|x\|^2$ which defines $S^2 = g^{-1}(\{1\})$. (Here, $T_x S^2$ is the equivalence classes of paths starting at x where two paths are equivalent if they have the same initial velocity; this initial velocity is an elt of $\ker T_x g$.)

Observe $T_x g = (2x_1, 2x_2, 2x_3)$ so (y_1, y_2, y_3) is in the kernel of $T_x g$ iff $\langle x, y \rangle = 0$. Hence

$$T_1 S^2 \cong \{ (x, y) : x \in S^2, y \in \mathbb{R}^3, \|y\| = 1, \langle x, y \rangle = 0 \} = M.$$



③ Let $\mathbb{R}P^n$ be real projective space,
realized as a topological space as $S^n/\sim$ where $x \sim -x$ and
 $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

- (a) Use covering spaces to compute $\pi_1(\mathbb{R}P^n)$.
- (b) Give a cell (CW) decomposition of $\mathbb{R}P^n$ for $n \geq 1$.
- (c) Use the cell decomposition to compute the homology groups $H_k(\mathbb{R}P^n)$, $k \geq 0$.
- (d) For which values of $n \geq 1$ is $\mathbb{R}P^n$ orientable? Explain.

solution to (a)

$$\mathbb{R}P^1 \cong S^1 \quad \text{so} \quad \pi_1(\mathbb{R}P^1) = \mathbb{Z}.$$

For $n \geq 2$, S^n is simply connected and is a covering space of $\mathbb{R}P^n$ w/ covering map $p: S^n \rightarrow \mathbb{R}P^n$ defined by $x \mapsto \{x, -x\}$.

so S^n is the universal cover of $\mathbb{R}P^n$.

$$\text{so } \pi_1(\mathbb{R}P^n) \cong \text{Aut}(S^n, p) := \left\{ \varphi: S^n \rightarrow S^n \text{ s.t. } p \circ \varphi = p \right\}_{\text{homeo}}$$

claim: $\varphi \in \text{Aut}(S^n, p)$ iff $\varphi(x) = x, \forall x \in S^n$ or $\varphi(x) = -x, \forall x \in S^n$.

Proof (claim): Define $f: S^n \rightarrow \mathbb{R}$ by $f(x) = +1$ if $\varphi(x) = x$ and -1 if $\varphi(x) = -x$. (nearly f is continuous. Then $S^n = f^{-1}(1) \cup f^{-1}(-1)$ is a decomp of S^n into two closed-open subspaces; since S^n is connected, one must be empty. $\checkmark$

we conclude $\text{Aut}(S^n, p) \cong \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{R}P^n), n \geq 2$.
 $\square$

solution to (b)

we can give it for $\mathbb{R}P^0$ too.

$$\mathbb{R}P^0 = \{\text{lines in } \mathbb{R}^1\} = \{*\} \text{ a point.}$$

$$n \geq 1: \mathbb{R}P^n = S^n / \sim \text{ where } x \sim -x \quad (S^n \subset \mathbb{R}^{n+1})$$

$$= D^n / \sim \text{ where } \begin{matrix} x \sim -x \\ x \in \partial D^n = S^{n-1} \end{matrix} \quad \left(\begin{matrix} D^n \subset S^n \\ \text{closed upper} \\ \text{hemisphere} \end{matrix} \right)$$

$$= D^n / \mathbb{R}P^{n-1}$$

Hence we see that $\mathbb{R}P^n$ has a cell decomposition $e^0 \cup e^1 \cup \dots \cup e^n$, where for $1 \leq i \leq n$ the i^{th} attaching map φ_i attaches the boundary S^{i-1} of $\bar{e}^i = D^i$ to $\mathbb{R}P^{i-1}$ by identifying antipodal points.

More rigorously, define $\mathbb{R}P^n$ inductively, where for $n \geq 1$, $\mathbb{R}P^n := \mathbb{R}P^{n-1} \cup_{\varphi_n} D^n$ where as a set $\mathbb{R}P^{n-1} = S^{n-1} / \sim$, $S^{n-1} = \partial D^n$, and where $\varphi_n: \partial D^n \rightarrow \mathbb{R}P^{n-1}$, $x \mapsto [x]$.
 $\square$

solution to (c)

$$\mathbb{R}P^n = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$$

where $X^{(k)} = X^{(k)} \amalg_{\varphi_k} B^{k+1}$

where B^{k+1} = closed unit ball in $\mathbb{R}^{k+1}$

and $\varphi_k : S^k = \partial B^{k+1} \rightarrow X^{(k)} \cong \mathbb{R}P^k$

is defined by $\varphi_k(x) = \phi_k(\{x, -x\})$.

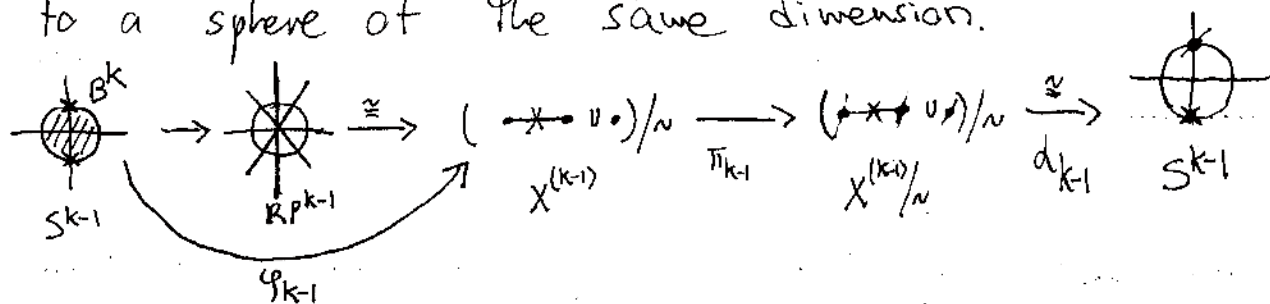
Let $\pi_k : X^{(k)} \rightarrow X^{(k)} \cong S^k$ be defined

by collapsing $X^{(k-1)}$ to a point.

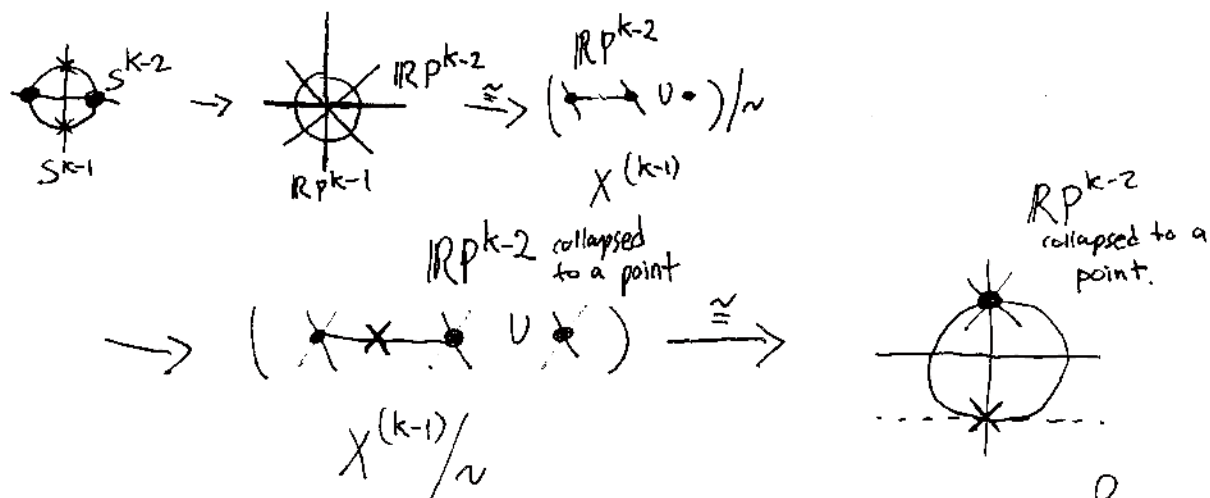
We wish to compute the degree of

$$S^{k-1} \xrightarrow{\varphi_{k-1}} X^{(k)} \xrightarrow{\pi_{k-1}} X^{(k)}/\sim \xrightarrow{\cong} S^{k-1}$$

where d_{k-1} is the standard iso sending a ball modulo its boundary to a sphere of the same dimension.



Let $S^{k-2} \subset S^{k-1} \subset \mathbb{R}^k$ be the set of points in S^{k-1} w/ zero last coordinate. The above pictures become



we see that the composite $S^{k-1} \xrightarrow{f_{k-1}} S^{k-1}$ stretches each (open) upper/lower hemisphere homeomorphically onto $S^{k-1} - \{\text{north? pole}\}$. Let f_{k-1}^N, f_{k-1}^S be the corresponding homeos.

Because the first map defining the composite $S^{k-1} \rightarrow S^{k-1}$ is $\phi^{S^{k-1}} \rightarrow \star RP^{k-1}$, clearly

$$f_{k-1}^N = f_{k-1}^S \circ \mathcal{A} \text{ or } f_{k-1}^S = f_{k-1}^N \circ \mathcal{A},$$

where $\mathcal{A}$ is the antipodal map $S^{k-1} \rightarrow S^{k-1}$, having degree $(-1)^{k-1}$ (restricting $\mathcal{A}$ to upper/lower (open) hemisphere, respectively.)

Let $[\sigma^N - \sigma^S]$ be the generator of $H_{k-1}(S^{k-1})$ where $\sigma^N/\sigma^S: \Delta^{k-1} \rightarrow S^{k-1}$ map Δ^{k-1} homeomorphically (rel boundary) to the closed upper/lower hemisphere, respectively.

$$\begin{aligned} \text{Then } H_{k-1}(S^{k-1} \xrightarrow{f} S^{k-1})([\sigma^N - \sigma^S]) \\ &= [f \circ \sigma^N - f \circ \sigma^S] \\ &= [f \circ \sigma^N - f \circ \iota \circ \sigma^N] \\ &= [f_{k-1}^N \circ \sigma^N] \end{aligned}$$

Let U_N and U_S be the open upper and lower hemispheres, respectively, and let V be the sphere minus the north pole. We want to compute the local degrees of f_{k-1} w.r.t. U_N and U_S . More specifically, observe that if $X_N = \text{north pole}$ and $X_S = y = \text{south pole}$, then $f^{-1}(y) = \{X_N, X_S\}$ (follow the green X 's in the diagrams). Then we wish to compute the local degree of $H_{k-1}(S^{k-1}) \cong H_{k-1}(U_N, U_N - X_N) \xrightarrow{H_{k-1}f_{k-1}} H_{k-1}(V, V - y) \cong H_{k-1}(S^{k-1})$ and

for the same w/ $U_N \leftrightarrow U_S$ and $X_N \leftrightarrow X_S$.

(For simplicity, for the time being let $n \equiv k-1$) (And write $f \equiv f_{k-1}$)

The iso $H_n(S^n) \cong H_n(U_N, U_N - X_N)$

is determined by the comm diagram (black only):

$$\begin{array}{ccccc}
 & & H_n(U_N, U_N - X_N) & \xrightarrow{H_n(f)} & H_n(V, V - Y) \\
 & \swarrow N & \downarrow & & \downarrow \cong \\
 H_n(S^n, S^n - X_N) & \xleftarrow{H_n(f)} & H_n(S^n, S^n - f^{-1}(Y)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n - Y) \\
 & \nwarrow \cong & \uparrow & & \uparrow \cong \\
 & & H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n)
 \end{array}$$

To find a generator for $H_n(U_N, U_N - X_N)$ we follow the generator $[\sigma_N - \sigma_S]$ for $H_n(S^n)$, where σ_N and σ_S are singular simplices mapping Δ^n homeomorphically to the closure of U_N and U_S in S^n , respectively, and in a natural way s.t.

boundaries are preserved: $\Delta^n \xrightarrow{\sigma_N} \bigcirc$
and σ_N 'matches' correctly w/ σ_S .

The map $H_n(S^n) \rightarrow H_n(S^n, S^n - X_N)$ generally sends $[c] \mapsto [c]$. And the map $\partial c = 0$ $c \in C_n(S^n)$

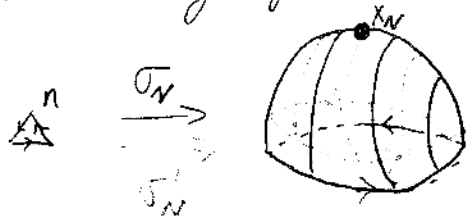
$H_n(U_N, U_N - X_N) \rightarrow H_n(S^n, S^n - X_N)$ generally sends $[c] \mapsto [c]$ $c \in C_n(U_N)$ $\partial c \in C_{n-1}(U_N - X_N)$

So $[\sigma_N - \sigma_S] \in H_n(S^n)$ is just mapped to $[\sigma_N - \sigma_S] \in H_n(S^n, S^n - X_N)$.

The harder part is finding the preimage of this elt in $H_n(U_N, U_N - X_N)$.

claim 1: Let $\sigma'_N \in C_n(U_N)$ be the singular simplex obtained by 'shrinking'

σ_N just slightly to lie in U_N :



Clearly $\partial\sigma'_N \in C_{n-1}(S^n - X_N)$, so

$[\sigma'_N]$ is a w.d. elt of $H_n(U_N, U_N - X_N)$.

The claim is $[\sigma'_N]$ is the preimage of $[\sigma_N - \sigma_S]$.

~~Proof (claim): we need to show~~

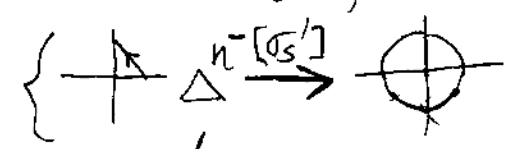
~~$[\sigma'_N] = [\sigma_N - \sigma_S] \in H_n(S^n, S^n - X_N)$, i.e.~~

Proof (claim):

Omitted. $\square$.

(The claim is at least intuitively plausible
— what else would generate $H_n(U_N, U_N - X_N)$?
— and I did validate the claim in
the case $n = 1^V$, but it involved
calculating w/ simplices which can be
complicated in higher dimensions.)

Cor/ $[\sigma'_N]$ generates $H_n(U_N, U_N - X_N)$.

Proof (cor): $\square$ 
 see claim 3!!!
below

Similarly, returning And see claim 4 for the def of σ'_S . to the (comm diagram above,
 $\{-[\sigma'_S]\}$ generates

$H_n(V, V - Y)$ and corresponds to the
generator $[\sigma'_N - \sigma'_S]$ of $H_n(S^n)$. So we
wish to calculate $H_n(f)([\sigma'_N])$ in terms of $-\sigma'_S$.

Regarding claim 1 when $n = 1$:

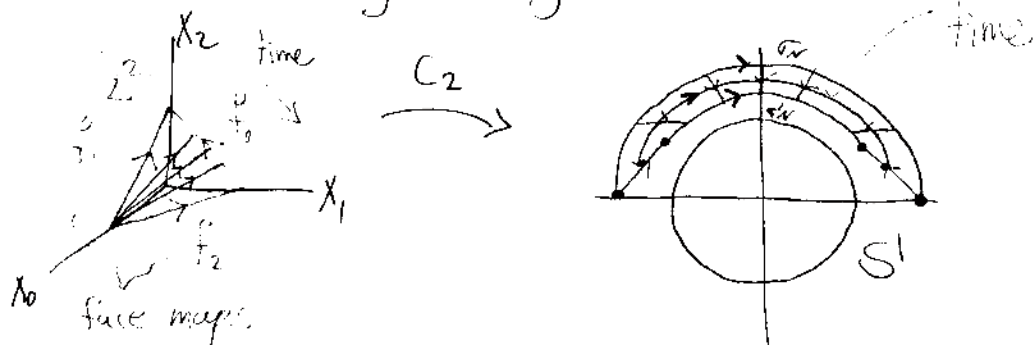
$$[\sigma'_N] = [\sigma_N - \sigma_S] \in H_n(S^n, S^n - X_N)$$

$$\text{iff } \sigma'_N - \sigma_N + \sigma_S \in B_n(S^n, S^n - X_N)$$

$$\text{iff There exists } c \in C_{n+1}(S^n) \text{ s.t.}$$

$$\partial C_{n+1} = \sigma'_N + \sigma_N - \sigma_S \in C_n(S^n - X_N)$$

When $n = 1$, we construct $C_2 \in C_2(S^1)$ as the singular 2-simplex indicated by the following diagram:



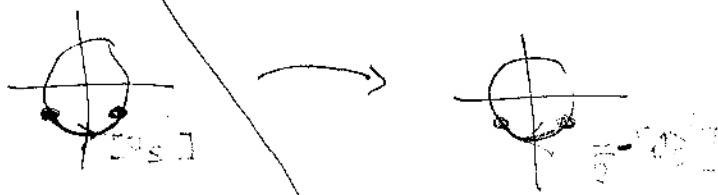
$$\text{One calculates } \partial C_2 = \underbrace{C_2 \circ f_0}_{\in C_1(S^1 - X_N)} - \sigma_N + \sigma'_N.$$

~~Claim 3 : $H_n(f)([\sigma'_S]) = (-1)^n [\sigma'_S] \in H_n(V-y)$~~

~~Proof (claim 3):~~

~~The proof is relatively clear when $n=1$:~~

~~Again, following the usual pictures we see~~



~~of the image $[\sigma]$. We wish to know~~

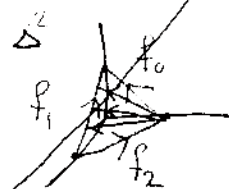
~~if $[\sigma] = -[\sigma'_S]$ (intuitively, yes)~~

~~iff $\sigma + \sigma'_S \in B_1(V, V-y)$~~

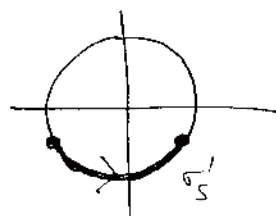
~~iff There exists $C_2 \in C_2(V)$ s.t.~~

~~$\partial C_2 - \sigma - \sigma'_S \in C_1(V-y)$~~

~~To construct C_2 , we use the same (almost) schematic as in the proof of claim 1:~~



$C_2 \rightarrow$

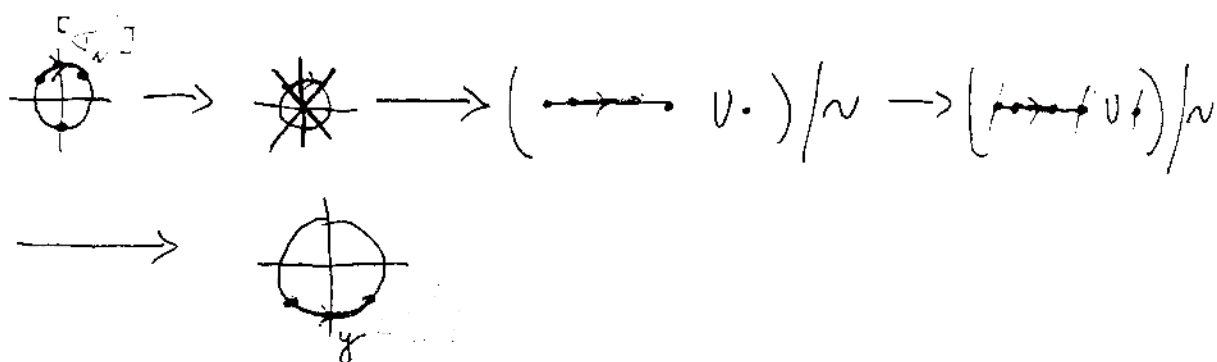


~~$(\partial C_2 = \sigma + \sigma'_S)$
'time' is irrelevant unlike before.~~

claim 2: $H_n(f)([\sigma'_n]) = [\sigma'_s] \in H_n(V, V-y)$

Proof (claim 2): $\left(\begin{array}{l} \text{recall we really mean } n \equiv k-1 \\ f \equiv f_{k-1} \end{array} \right)$

This is clear when $n = 1$:



we omit the proof for $n > 1$. $\square$

Hence the local degree $\deg f|_{X_N} = 1$.

The calculation of $\deg f|_{X_S}$ proceeds similarly:
 we wish to know where $H_n(f)$ takes
 the generator $[\sigma'_s]$ of $H_n(U_S, U_S - X_S)$
 (corresponding to $[\sigma'_s] \in H_n(S^n)$) in terms of the
 generator $[\sigma'_s]$ of $H_n(V, V-y)$
 (also corresponding to $[\sigma'_n - \sigma'_s] \in H_n(S^n)$, same as before). :

claim 3: $-\sigma'_S$ corresponds to $[\sigma_n - \sigma_S]$

in the isomorphism $H_n(U_S, U_S - X_S) \cong H_n(S^n)$

(or $H_n(V, V - y) \cong H_n(S^n)$)

Proof (claim 3):

The proof

when $n=1$:

we

show if σ is the 1-simplex  then

$$-\sigma'_S = [\sigma] \in H_1(U_S, U_S - X_S)$$

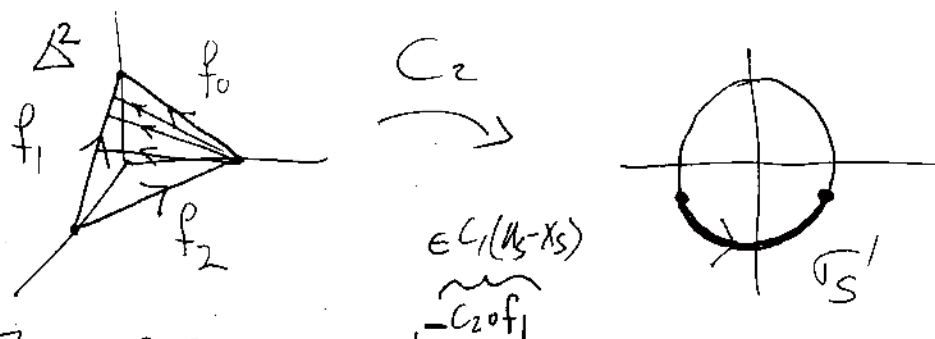
$$\text{iff } \sigma + \sigma'_S \in B_1(U_S, U_S - X_S)$$

$$\text{iff there exists } C_2 \in C_2(U_S) \text{ s.t.}$$

$$2C_2 - \sigma - \sigma'_S \in C_1(U_S - X_S)$$

We proceed using the same (almost)

schematic as in the proof of Claim 1:

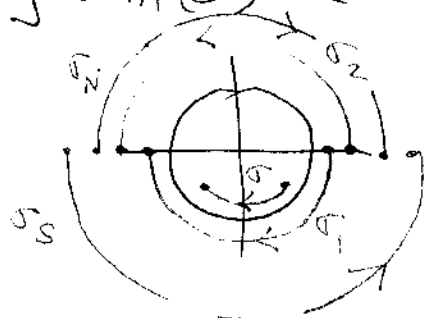


Then $2C_2 = \sigma + \sigma'_S$. note 'time' is irrelevant, unlike before.

From claim 2, it follows that

$$[\sigma_1 - \sigma_2] \in H_1(S') \iff [\sigma] \in H_1(U_S, U_S - X_S) \stackrel{=}{=} [\sigma'_S]$$

where:



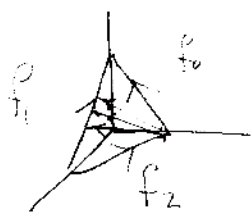
(turn the paper upside down)

we will be done if $[\sigma_1 - \sigma_2] = [\sigma_N - \sigma_S] \in H_1(S')$.

That is, $\sigma_1 - \sigma_2 - \sigma_N + \sigma_S$ is the boundary ∂C_2 of some $C_2 \in C_2(S')$

difference

Indeed, similar to before, take C_2 as the V of two simplices:



boundary

$$\sigma_S - C_B + \sigma_1$$

(C_B is the constant map at B)



minus



boundary

$$\sigma_N - C_B + \sigma_2$$

we omit the proof of claim 3 for $n > 1$.

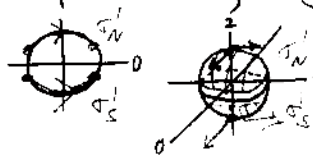
(The proof for $H_n(V, \mathbb{Z})$ is clearly identical.)



claim 4: (Geometric description of $-[\sigma'_S]$)

Define $z : S^n \rightarrow S^n$ to be the
 sending the last coordinate to its negative
 map. Then, all along in this solution, $\sigma'_S := z \circ \sigma'_N$

where σ'_N is understood.

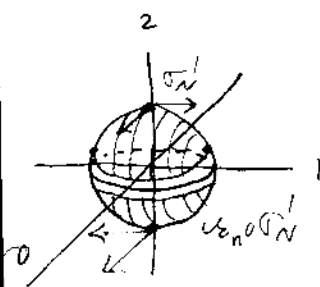
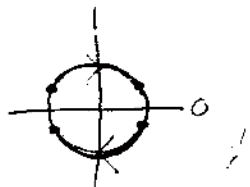


Define $\iota_n : S^n \rightarrow S^n$ by

$$\iota_n(x_0, x_1, \dots, x_n) = ((-1)^n x_0, -x_1, -x_2, \dots, -x_n).$$

// antipodal map $S^n \rightarrow S^n$

(So $\iota_n = \iota$ when n is odd.)



$$\text{Then } -[\sigma'_S] = [\iota_n \circ \sigma'_N] \in$$

$$H_n(U_S, U_S - X_S) \text{ or } \in H_n(V, V - Y).$$

Moreover, the same is true if ι_n is
 replaced by $\iota_n(\epsilon_0, \epsilon_1, \dots, \epsilon_{n-1}) : S^n \rightarrow S^n$ defined
 by $(x_0, x_1, \dots, x_{n-1}, x_n) \mapsto (\epsilon_0 x_0, \epsilon_1 x_1, \dots, \epsilon_{n-1} x_{n-1}, -x_n)$
 where $\epsilon_i = \pm 1$ and $\#\{i : \epsilon_i = -1\}$ is odd.

Proof

The proof for $n=1$ was done in
 the proof of claim 3. we omit the
 proof for $n>1$. $\square$

The geometric content of claim 2 is that under f , σ_N' maps to σ_S' . The consequence, by claim 3, is that the local degree is -1 .

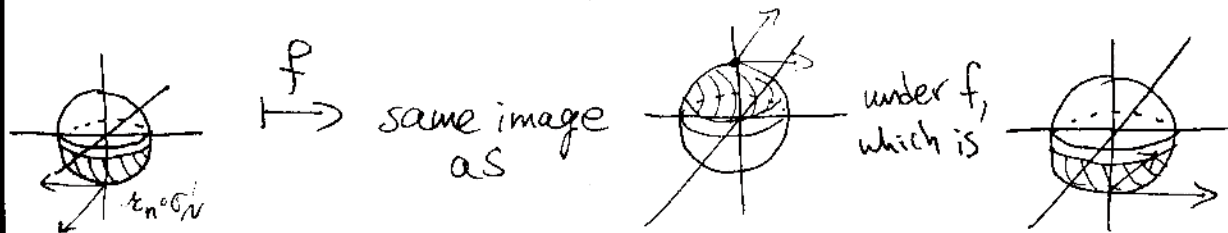
Observe that if n is odd, then by the definition of f , $\tau_n \circ \sigma_N' = \tau_n \circ \sigma_N'$ is sent to the same image as σ_N' ,

$$\text{hence } H_n^{\text{tr}}(-[\sigma_S']) = H_n^{\text{tr}}([\tau_n \circ \sigma_N'])$$

$$= H_n^{\text{tr}}([\sigma_N']) = [\sigma_S'] \in H_n(V, V-y),$$

by claims 2 & 4. Hence the local degree is again -1 . Conversely,

claim 5 : let n be even. f maps $\tau_n \circ \sigma_N'$ to $\tau_n(-1, +1, \dots, +1) \circ \sigma_N'$.



Proof: Obvious. $\square$.

Consequently, when n is even,
by claim 4, $H_n(f)(-[\sigma'_s]) = -[\sigma'_s]$

Hence the local degree is $+1$. $\in H_n(V, V-y)$.

Summarizing: The degree of f is (sum local degrees) = $-1 + (-1)$ when n is odd and $-1 + (+1)$ when n is even. Recall now $n \equiv k-1$ and $f \equiv f_{k-1}$.

By the main theorem of cellular homology we obtain the cellular complex for $\mathbb{R}P^n$

$$\cdots \rightarrow H_k(\mathbb{Z}^{(k)}, \mathbb{Z}^{(k+1)}) \xrightarrow{d_k} H_{k-1}(\mathbb{Z}^{(k-1)}, \mathbb{Z}^{(k-2)}) \rightarrow \cdots$$

where d_k is ^{multiplication by} the degree of f_{k-1} . So:

$$n \text{ odd: } \mathbb{Z}^0 \xrightarrow{n} \mathbb{Z}^{-2} \xrightarrow{n-2} \mathbb{Z}^0 \xrightarrow{\cdots} \mathbb{Z}^{-2} \xrightarrow{3} \mathbb{Z}^0 \xrightarrow{2} \mathbb{Z}^{-2} \xrightarrow{1} \mathbb{Z}^0 \rightarrow 0$$

$$n \text{ even: } 0 \rightarrow \mathbb{Z}^{-2} \xrightarrow{n-2} \mathbb{Z}^0 \xrightarrow{n-2} \mathbb{Z}^{-2} \xrightarrow{\cdots} \mathbb{Z}^{-2} \xrightarrow{3} \mathbb{Z}^0 \xrightarrow{2} \mathbb{Z}^{-2} \xrightarrow{1} \mathbb{Z}^0 \rightarrow 0$$

$$\text{So } H_k(\mathbb{R}P^n) = \begin{cases} \text{odd: } \mathbb{Z} (k=0, n), \mathbb{Z}_2 (k=1, 3, \dots, n-2), 0 (k=2, 4, \dots, n-1) \\ \text{even: } \mathbb{Z} (k=0), \mathbb{Z}_2 (k=1, 3, \dots, n-1), 0 (k=2, 4, \dots, n). \end{cases} \square$$

solution to (d)

We want to answer the question

When is $\mathbb{R}P^n$ orientable?

topological
mfd of dim n
↓

Since $\mathbb{R}P^n$ is a compact and connected,

{ if orientable, then $H_n(\mathbb{R}P^n) = \mathbb{Z}$
if not orientable, then $H_n(\mathbb{R}P^n) = 0$.

we conclude $\mathbb{R}P^n$ is orientable when
 n is odd, and is not orientable when
 n is even.

sanity check: $\mathbb{R}P^1 \cong S^1$ orientable. ✓

$\mathbb{R}P^2$ is not orientable.



(4) Let $f: X \rightarrow Y$ be a continuous map between top spaces. Define

$$C_f = ((X \times [0,1]) \amalg Y) / \sim$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and

$(x, 0) \sim (x', 0)$ for all $x, x' \in X$.

Prove there is a LES

$$\dots \rightarrow H_{k+1}(X) \xrightarrow{H_{k+1}(f)} H_{k+1}(Y) \xrightarrow{\sim} \tilde{H}_{k+1}(C_f) \xrightarrow{H_k(f)} H_k(X) \rightarrow H_k(Y) \rightarrow$$

where $\tilde{H}_k(C_f)$ is the k^{th} reduced homology group.

~~Solution~~

~~Let $X' := f(X) \subset Y$. Then we have a~~

~~LES in relative homology~~

$$\xrightarrow{\delta_{k+2}} H_{k+2}(Y, X') \xrightarrow{\delta_{k+1}} H_{k+1}(X') \xrightarrow{H_{k+1}(i)} H_{k+1}(Y) \xrightarrow{\delta_{k+1}} H_{k+1}(Y, X') \xrightarrow{\delta_k} H_k(X') \xrightarrow{H_k(i)} H_k(Y) \rightarrow$$

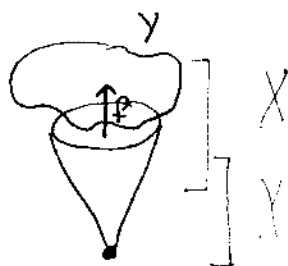
$$\delta_{k+1}([c]) = [2c].$$

$$2c \in C_k(X)$$

solution

Let $X_2 = ((X \times [\frac{1}{4}, 1]) \sqcup Y) / \sim$

and $X_1 = (X \times [0, \frac{3}{4}]) / \sim$.



Clearly $X_1 \cap X_2$ is a

nbhd def retract in X_1 and X_2 ,

and $X_1, X_2 \subset C_p$ are closed.

So we get a Mayer-Vietoris LES

$$\rightarrow H_k(X_1 \cap X_2) \rightarrow H_k(X_1) \oplus H_k(X_2) \rightarrow H_k(C_p) \rightarrow H_{k-1}(X_1 \cap X_2) \rightarrow$$

Clearly $H_k(X_1 \cap X_2) \cong H_k(X)$ and

$$H_k(X_1) \oplus H_k(X_2) \cong \begin{cases} H_k(Y), & k > 0 \\ \mathbb{Z} \oplus H_0(Y), & k = 0 \end{cases}$$

Since everything is a def retract.

we need to make sense of these iso's
as well as the claim $H_k(X) \rightarrow H_k(Y)$ is the map $H_k(p)$.

If ^{a subspace} A is a deformation retract of a space X , then there is an isomorphism $H_k(A) \xrightarrow{\sim} H_k(X)$ which is induced by the embedding $A \xrightarrow{i} X$ and so has the formula

$$[\sigma] \mapsto [i_* \sigma] \in H_k(X).$$

$\sigma \in C_k(A) \text{ simplex}$
 $\partial \sigma = 0$

Conversely, if $X \xrightarrow{p} A$ is the retraction at the end of the def retract, then there is an iso $H_k(X) \xrightarrow{\sim} H_k(A)$ induced by p and so has the formula

$$[\sigma] \mapsto [p_* \sigma].$$

$\sigma \in C_k(X) \text{ simplex}$
 $\partial \sigma = 0$

Identify X w/ $X \times \{\frac{1}{2}\}$ and let p_k be the retract $X_1 \cap X_2 \xrightarrow{p_k} X$ defined by $(x, t) \xrightarrow{p_k} (x, \frac{1}{2})$. So we have an iso $H_k(X_1 \cap X_2) \xrightarrow{H_k(p_k)} H_k(X)$. Let i_k be the embedding $X \rightarrow X_1 \cap X_2$ inducing an iso in the reverse direction.

Let q_k be the retraction $X_2 \rightarrow Y$ defined by $(x, t) \xrightarrow{q_k} f(x)$ and $y \xrightarrow{q_k} y$, inducing an iso $H_k(X_2) \xrightarrow{H_k(q_k)} H_k(Y)$. Let j_k be the reverse inclusion $Y \xrightarrow{j_k} X_2$.

Let Γ_K be the retraction $X_1 \rightarrow \{p\}$,

where p is the point $X \times \{0\} \in C_f$, defined by
 $(x, t) \mapsto (x, 0) = p$, which induces an iso

$H_K(X_1) \xrightarrow{H_K(\Gamma_K)} H_K(p)$. Let ι_K be the reverse inclusion
 $p \xrightarrow{\iota_K} X_1$.

Let a_1 and a_2 be the inclusions $X_1 \cap X_2 \xrightarrow{a_1} X_1$
 and $X_1 \cap X_2 \xrightarrow{a_2} X_2$. Let b_1 and b_2 be the inclusions
 $X_1 \xrightarrow{b_1} C_f$ and $X_2 \xrightarrow{b_2} C_f$. Recall the

Mayer-Vietoris maps are given by

$$\rightarrow H_K(X_1 \cap X_2) \xrightarrow{(-H_K(a_1)) \oplus (+H_K(a_2))} H_K(X_1) \oplus H_K(X_2) \xrightarrow{H_K(b_1) + H_K(b_2)} H_K(C_f) \xrightarrow{\sim} H_{K-1}(X_1 \cap X_2)$$

Using these various maps, we may construct a sequence

$$\begin{array}{ccccccc} H_{K+1}(C_f) & \rightarrow & H_K(X_1 \cap X_2) & \rightarrow & H_K(X_1) \oplus H_K(X_2) & \rightarrow & H_K(C_f) \rightarrow \\ & \searrow & \downarrow \uparrow & & \downarrow \uparrow & \nearrow & \searrow \\ & & H_K(X) & \rightarrow & H_K(p) \oplus H_K(Y) & & \end{array}$$

from which we get

$\rightarrow H_{K+1}(C_f) \rightarrow H_K(X) \rightarrow H_K(p) \oplus H_K(Y) \rightarrow H_K(C_f) \rightarrow$ which is clearly exact

since we have done nothing. Now, when $K > 0$, $H_K(p) \equiv 0$, so

the map $H_K(X) \rightarrow H_K(p) \oplus H_K(Y) \equiv H_K(Y)$ is given by the

composition $H_K(X) \xrightarrow{H_K(\iota_K)} H_K(X_1 \cap X_2) \xrightarrow{(-H_K(a_1)) \oplus (+H_K(a_2))} H_K(X_1) \oplus H_K(X_2) \xrightarrow{H_K(\Gamma_K) \oplus H_K(\iota_K)} H_K(p) \oplus H_K(Y) \equiv H_K(Y)$

given by the formula

$$[\sigma] \mapsto [i_k \circ \sigma] \mapsto (-[a_1 \circ i_k \circ \sigma], [a_2 \circ i_k \circ \sigma]) \mapsto [z_k \circ a_2 \circ i_k \circ \sigma]$$

$\sigma \in C_k(X)$
 $\partial \sigma = 0$

And the simplex $z_k \circ a_2 \circ i_k \circ \sigma$ has the formula

$$(X, \frac{1}{2}) \mapsto (X, \frac{1}{2}) \mapsto (X, \frac{1}{2}) \mapsto f(X)$$

So we see the map $H_k(X) \rightarrow H_k(Y)$ is indeed the map $H_k(f)$ for $k > 0$.

When $k = 0$, we have the part of the LES

$$\rightarrow H_0(X) \rightarrow \overbrace{H_0(p)}^{\cong \mathbb{Z}} \oplus H_0(Y) \rightarrow H_0(Cf) \rightarrow 0 \rightarrow$$

We want the end of our LES to look like

$$\rightarrow H_0(X) \rightarrow H_0(Y) \rightarrow \underbrace{\tilde{H}_0(Cf)}_{\cong \mathbb{Z}^{(\text{rank } H_0(Cf) - 1)}} \rightarrow 0 \rightarrow$$

Let r be the number of components of X so that $\{[X_i]_{i=1, \dots, r}\}$ is a basis of $H_0(X)$ for any $x_i \in X_i$ the i th component of X . The map $H_0(X) \rightarrow H_0(p) \oplus H_0(Y)$ is given by $\sum_{i=1}^r n_i [X_i] \mapsto (-\sum_{i=1}^r n_i [p], \sum_{i=1}^r n_i [f(x_i)])$.

(Let $y_0 \in Y$ be in a component containing a piece of the image $f(X)$. Then since $(X \times [0, 1])_{\sim}$ is path connected in Cf , we may

re write the above map $H_0(X) \rightarrow H_0(p) \oplus H_0(Y)$ as

$$\sum_{i=1}^{r_X} n_i [X_i] \mapsto \left(-\sum_{i=1}^{r_X} n_i [p], \sum_{i=1}^{r_X} n_i [y_0] \right)$$

write $[f(X)] = [y_0] \in H_0(C_p)$ for all $X \in X$.
 This parenthetical comment will have relevance shortly.)

Let r_Y be the number of components of Y .
 For any $y_i \in Y_i$, the i^{th} component of Y , $i=1, \dots, r_Y$,
 $\{[y_i]\}_{i=1, \dots, r_Y}$ is a basis for $H_0(Y)$. Let $X_i, i=1, \dots, r_X$,
 be as before. For each $i=1, \dots, r_Y$ let $A_i = \{X_j : j=1, \dots, r_X \text{ and } f(X_j) \in Y_i\}$. Then the map
 $H_0(X) \rightarrow H_0(p) \oplus H_0(Y)$ from before has the formula
 $\sum_{j=1}^{r_X} n_j [X_j] \mapsto \left(-\sum_{j=1}^{r_X} n_j [p], \sum_{i=1}^{r_Y} \left(\sum_{j \in A_i} n_j \right) [y_i] \right)$. This is
 the same as the function $\mathbb{Z}^{r_X} \xrightarrow{F} \mathbb{Z} \oplus \mathbb{Z}^{r_Y}$ defined by
 $(n_1, \dots, n_{r_X}) \xrightarrow{F} \left(-\sum_{j=1}^{r_X} n_j, \sum_{j \in A_1} n_j, \dots, \sum_{j \in A_{r_Y}} n_j \right)$.

Next, the map $H_0(p) \oplus H_0(Y) \rightarrow H_0(C_p)$ is
 given by $(m[p], \sum_{i=1}^{r_Y} m_i [y_i]) \mapsto m[p] + \sum_{i=1}^{r_Y} m_i [y_i]$
 $= (m + \sum_{i=1}^{r_Y} m_i) [y_0] + \sum_{i=\tilde{r}_Y+1}^{r_Y} m_i [y_i]$, where $y_0 \in Y$ is
 as above in the parenthetical comment, and where $\tilde{r}_Y$

is... such that the image $f(X)$ intersects Y_i for $i=1, \dots, \tilde{r}_Y$ but not $i=\tilde{r}_Y+1, \dots, r_Y$. So, this map is the same as the map $\mathbb{Z} \oplus \mathbb{Z}^{r_Y} \xrightarrow{G} \mathbb{Z} \oplus \mathbb{Z}^{r_Y - \tilde{r}_Y}$ defined by $(m, m_1, \dots, m_{r_Y}) \mapsto (m+m_1+\dots+m_{\tilde{r}_Y}, m_{\tilde{r}_Y+1}, \dots, m_{r_Y})$.

Observe we have made the old LES look at the end like

$$\begin{array}{ccccccc} \varepsilon & \rightarrow & \underbrace{\mathbb{Z}^{r_X}}_{\cong H_0(X)} & \xrightarrow{F} & \mathbb{Z} \oplus \underbrace{\mathbb{Z}^{r_Y}}_{\cong H_0(Y)} & \xrightarrow{G} & \mathbb{Z} \oplus \underbrace{\mathbb{Z}^{r_Y - \tilde{r}_Y}}_{\cong \tilde{H}_0(Y)} \rightarrow 0 \rightarrow \end{array}$$

We augment this ^{in the obvious way} to obtain the new tail

$$\varepsilon \rightarrow \mathbb{Z}^{r_X} \xrightarrow{F'} \mathbb{Z}^{r_Y} \xrightarrow{G'} \mathbb{Z}^{r_Y - \tilde{r}_Y} \rightarrow 0 \rightarrow$$

Specifically, F' is defined by $(n_1, \dots, n_{r_X}) \mapsto \left(\sum_{j \in A_1} n_j, \dots, \sum_{j \in A_{r_Y}} n_j \right)$

and G' is defined by $(m_1, \dots, m_{r_Y}) \mapsto (m_{\tilde{r}_Y+1}, \dots, m_{r_Y})$
(compare w/ the def's of F and G). Note: If $\tilde{r}_Y = r_Y$, then $\mathbb{Z}^{r_Y - \tilde{r}_Y} \cong \tilde{H}_0(Y) \cong 0$ and $G' = 0$.

If the resulting LES remains exact, then we are done.

Exactness at $\mathbb{Z}^{r_X}$: By assumption $\text{im } \varepsilon = \ker F$.

So we must show $\ker F = \ker F'$. Indeed, $\ker F =$

$$\left\{ (n_1, \dots, n_{r_X}) : -\sum_{j=1}^{r_X} n_j = \sum_{j \in A_1} n_j = \dots = \sum_{j \in A_{r_Y}} n_j = 0 \right\} = \left\{ (n_1, \dots, n_{r_X}) : \sum_{j \in A_1} n_j = \dots = \sum_{j \in A_{r_Y}} n_j \right\} = \ker F'$$

Exactness at $\mathbb{Z}^{\tilde{r}_Y}$: For concreteness, we divide into the cases $\tilde{r}_Y = r_Y$ and $\tilde{r}_Y \neq r_Y$. (i) If $\tilde{r}_Y = r_Y$, then $G' \equiv 0$ and $\ker G' = \mathbb{Z}^{\tilde{r}_Y}$. Recall A_i consists of those $j = 1, \dots, r_X$ s.t. $f(X_j) \in Y_i$. If $\tilde{r}_Y = r_Y$, then all $A_i \neq \emptyset$, so F' is clearly surjective. (ii) If $\tilde{r}_Y \neq r_Y$, then some $A_i = \emptyset$ ($i = \tilde{r}_Y + 1, \dots, r_Y$), so F' surjects onto $\mathbb{Z} \times \dots \times \mathbb{Z} \times \underset{\substack{\uparrow \\ \tilde{r}_Y + 1}}{0} \times \dots \times 0$, and the kernel of G' is also those $(m_1, \dots, m_{\tilde{r}_Y}, m_{\tilde{r}_Y + 1}, \dots, m_{r_Y})$ whose last $r_Y - \tilde{r}_Y$ coordinates vanish, hence $\text{im } F' = \ker G'$.

Exactness at $\mathbb{Z}^{r_Y - \tilde{r}_Y}$: This is immediate since G' is clearly surjective.

This completes the solution. $\square$.

(5) A Lie group G is a smooth manifold that is also a group s.t. the group multiplication $G \times G \rightarrow G$ and inversion $G \rightarrow G$ operations are smooth. Prove the fundamental group of a connected Lie group is abelian.

solution

Since G is connected, we may assume our loops are based at the identity.

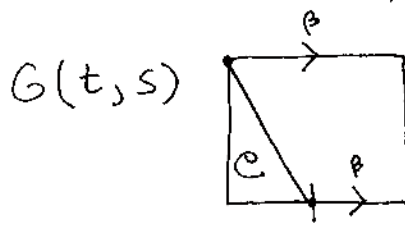
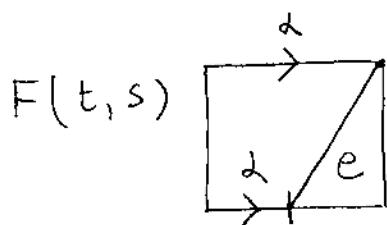
Let $\alpha(t)$ and $\beta(t)$ be loops.

By $(\alpha * \beta)(t)$ it is meant the

concatenation $\xrightarrow{\alpha} + \xrightarrow{\beta}$, whereas $\alpha(t) \cdot \beta(t)$ refers to the product of the paths in the group G .

We show $(\alpha * \beta)(t) \simeq \alpha(t) \cdot \beta(t) \simeq (\beta * \alpha)(t)$

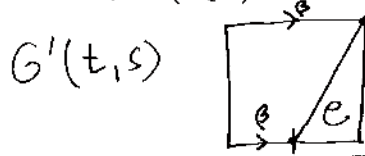
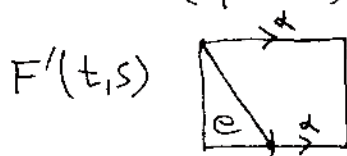
Let $F(t,s)$ and $G(t,s)$ be the path homotopies



Then $F(t,s) \cdot G(t,s)$ is the desired path homotopy

from $(\alpha * \beta)(t)$ to $\alpha(t) \cdot \beta(t)$.

And $(\beta * \alpha)(t) \simeq \alpha(t) \cdot \beta(t)$ via $F'(t,s) \cdot G'(t,s)$ where



⑥ $M \subset \mathbb{R}^3$ embedded compact oriented surface w/out boundary of genus ≥ 1 .

Prove the Gaussian curvature must vanish somewhere.

solution

Since M is compact embedded, it has a point of positive curvature.

Since M is of genus $g \geq 1$, and oriented, its Euler characteristic is $2 - 2g \leq 0$.

By Gauss-Bonnet, $\int_M K = 2\pi \chi(M) \leq 0$.

so it must have a point of ≤ 0 curvature.

By the I V T, it has a point of 0 curvature.

