2011, Spring

Problem 1.

We have $\mathsf{d}\omega = \mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3 \wedge \mathsf{d}x_4$ by a simple computation, and so

$$\int_{\mathsf{S}^3} \omega = \int_{\mathsf{B}^4} \mathsf{d}\omega = \int_{\mathsf{B}^4} \mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3 \wedge \mathsf{d}x_4 = \mathsf{vol}(\mathsf{B}^4)$$

by Stokes. □

Problem 2.

- Consider the smooth map $f : \mathbb{R}^6 \to \mathbb{R}^3$ given by

$$f(x,y) := (\underbrace{x_1^2 + x_2^2 + x_3^2}_{=\|x\|^2}, \underbrace{y_1^2 + y_2^2 + y_3^2}_{=\|y\|^2}, \underbrace{x_1y_1 + x_2y_2 + x_3y_3}_{=\langle x,y\rangle}).$$

  Then $M = f^{-1}(1,1,0)$ by definition. For any $(x,y) \in f^{-1}(1,1,0)$, consider the differential

$$\mathsf{d}f_{(x,y)} = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2y_1 & 2y_2 & 2y_3 \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{pmatrix}$$

  Now there must be $1 \le i,j \le 3$ such that $x_i, y_j \neq 0$ (since $\|x\|, \|y\| = 1$) with $i \neq j$ (since $\langle x,y\rangle = 0$). Then the first row is nonzero in the i-th column, the second row in the $(3+j)$-th column, and the last row in the j-th and $(3+i)$-th columns. Thus $(1,1,0)$ is a regular value of f, whereby M is an embedded 3-dimensional submanifold of $\mathbb{R}^6$.

- Since f is continuous, the preimage M of the closed point $(1,1,0)$ is closed. So to see that M is compact, it remains to check that M is bounded. But this is immediate since for any $(x,y) \in M$, we have $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2 = 2$.

- Let $u : \mathbb{R}^3 \to \mathbb{R}$ be given by $u(x) := \|x\|^2$, so that $\mathsf{S}^2 = u^{-1}(1)$. Then at each point $x \in \mathbb{R}^3$, upon canonically identifying $\mathsf{T}_x\mathbb{R}^3 \cong \mathbb{R}^3$, we get

$$\mathsf{T}_x\mathsf{S}^2 = \ker(\mathsf{d}u_x) = \ker\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \end{pmatrix} = \{y \in \mathbb{R}^3 \mid \langle x,y\rangle = 0\},$$

  and so

$$M = \{(x,y) \in \mathbb{R}^6 \mid \|x\| = 1, \|y\| = 1, \langle x,y\rangle = 0\} \cong \{(x,y) \mid x \in \mathsf{S}^2, y \in \mathsf{T}_x\mathsf{S}^2, \|y\| = 1\}.$$

  The right-hand side is precisely the definition of the unit tangent bundle of S^2.

□

Problem 3.

(a) Firstly, $\mathbb{R}\mathsf{P}^1 \cong \mathsf{S}^1$ and so $\pi_1(\mathbb{R}\mathsf{P}^1) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$. If now $n \ge 2$, then $\mathbb{R}\mathsf{P}^n$ is the quotient of S^n by the antipodal action of $\mathbb{Z}_2$ defined by $1 \cdot x := x$ and $-1 \cdot x := -x$ for all $x \in \mathsf{S}^n$. Then $\mathbb{Z}_2$ is the group of deck transformations of the (normal, simply connected) universal cover $\mathsf{S}^n \twoheadrightarrow \mathbb{R}\mathsf{P}^n$, and so $\pi_1(\mathbb{R}\mathsf{P}^n) \cong \mathbb{Z}_2$. □

(b) We may construct S^n by starting with two 0-cells e^0_-, e^0_+, then gluing on two "half-circle" 1-cells e^1_-, e^1_+, then gluing on two "half-sphere" 2-cells e^2_-, e^2_+, etc., until we've glued on two "half-sphere" n-cells e^n_-, e^n_+. In the quotient $\mathbb{R}\mathsf{P}^n = \mathsf{S}^n/\mathbb{Z}_2$, we identify e^j_- and e^j_+ for each $0 \le j \le n$. Thus $\mathbb{R}\mathsf{P}^n$ consists of exactly one j-cell e^j (with attaching map the 2-fold cover $p_{j-1} : \mathsf{S}^{j-1} \twoheadrightarrow \mathbb{R}\mathsf{P}^{j-1}$) for each $0 \le j \le n$. □

(c) By (b), the cellular chain complex $(\mathsf{C}^{\mathsf{CW}}_\bullet(\mathbb{R}\mathsf{P}^n), \partial_\bullet)$ of $\mathbb{R}\mathsf{P}^n$ is given by

$$0 \longrightarrow \mathbb{Z}\langle e^n \rangle \xrightarrow{\partial_n} \mathbb{Z}\left\langle e^{n-1} \right\rangle \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbb{Z}\left\langle e^1 \right\rangle \xrightarrow{\partial_1} \mathbb{Z}\left\langle e^0 \right\rangle \longrightarrow 0.$$

Now fix some $1 \le j \le n$ and recall that the boundary map $\partial_j : \mathsf{C}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \to \mathsf{C}^{\mathsf{CW}}_{j-1}(\mathbb{R}\mathsf{P}^n)$ is given by $\partial_j(e^j) = \mathsf{deg}(q_{j-1} \circ p_{j-1})e^{j-1}$, where q_{j-1} is the natural quotient map in the diagram

$$\mathsf{S}^{j-1} \xrightarrow{p_{j-1}} \mathbb{R}\mathsf{P}^{j-1} \xrightarrow{q_{j-1}} \mathbb{R}\mathsf{P}^{j-1}/\mathbb{R}\mathsf{P}^{j-2} \cong \mathsf{S}^{j-1}.$$

The restriction maps $q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-}, q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+}$ are homeomorphisms from the two hemispheres $e^{j-1}_-, e^{j-1}_+ \subset \mathsf{S}^{j-1}$, respectively, onto the space $\mathbb{R}\mathsf{P}^{j-1} \setminus \mathbb{R}\mathsf{P}^{j-2}$. Furthermore, letting $a : \mathsf{S}^{j-1} \to \mathsf{S}^{j-1}$ be the degree-$(-1)^j$ antipodal map, we have that

$$q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-} = q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+} \circ a,$$

and so

$$\mathsf{deg}(q_{j-1} \circ p_{j-1}) = \mathsf{deg}\left(q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_-}\right) + \mathsf{deg}\left(q_{j-1} \circ p_{j-1}\big|_{e^{j-1}_+}\right) = (-1)^j + 1 = \begin{cases} 0 & j \text{ odd}, \\ 2 & j \text{ even}. \end{cases}$$

Thus if n is odd or even, then the cellular chain complex of $\mathbb{R}\mathsf{P}^n$ is given by

$$0 \longrightarrow \mathbb{Z}\langle e^n \rangle \xrightarrow{0} \mathbb{Z}\left\langle e^{n-1} \right\rangle \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z}\left\langle e^1 \right\rangle \xrightarrow{0} \mathbb{Z}\left\langle e^0 \right\rangle \longrightarrow 0$$

or

$$0 \longrightarrow \mathbb{Z}\langle e^n \rangle \xrightarrow{2} \mathbb{Z}\left\langle e^{n-1} \right\rangle \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z}\left\langle e^1 \right\rangle \xrightarrow{0} \mathbb{Z}\left\langle e^0 \right\rangle \longrightarrow 0,$$

respectively. From the sequences above, for $0 < j < n$,

$$\mathsf{H}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \cong \frac{\mathsf{ker}(\partial_j)}{\mathsf{im}(\partial_{j+1})} \cong \begin{cases} \mathsf{ker}(0)/\mathsf{im}(2) & j \text{ odd}, \\ \mathsf{ker}(2)/\mathsf{im}(0) & j \text{ even} \end{cases} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & j \text{ odd}, \\ 0 & j \text{ even}. \end{cases}$$

By path connectedness, $\mathsf{H}^{\mathsf{CW}}_0(\mathbb{R}\mathsf{P}^n) \cong \mathbb{Z}$. And,

$$\mathsf{H}^{\mathsf{CW}}_n(\mathbb{R}\mathsf{P}^n) \cong \mathsf{ker}(\partial_n) \cong \begin{cases} \mathbb{Z} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}$$

It's clear that $\mathsf{H}^{\mathsf{CW}}_j(\mathbb{R}\mathsf{P}^n) \cong 0$ for all $j > n$. □

(d) $\mathbb{R}\mathsf{P}^n$ **is orientable if and only if** $n \geq 1$ **is odd**. Recall that a compact connected oriented (topological) n-manifold X without boundary has $\mathsf{H}_n(X;\mathbb{Z}) \cong \mathbb{Z}$. So by (c), $\mathbb{R}\mathsf{P}^n$ is unorientable if n is even. If n is odd, then a choice of connected component of $\mathsf{H}_n(\mathbb{R}\mathsf{P}^n;\mathbb{Z})\setminus 0 \cong \mathbb{Z}\setminus 0$ specifies an orientation on $\mathbb{R}\mathsf{P}^n$. ☐

Problem 4.

See problem 5 of 2013, Fall, replacing g by f, and replacing f by a constant map.

Problem 5.

Remark. The argument below actually works for G an arbitrary connected topological group.

Since G is connected, it suffices to show that $\pi_1(G,1)$ is abelian. We're done if we can find, for any pair of loops $f, g : [0,1] \to G$ based at 1, a homotopy between $f \cdot g$ and $g \cdot f$, where $\cdot$ is the product in G. Consider the families of maps $\{u_t : [0,1] \to G\}_{0 \leq t \leq 1}$ and $\{v_t : [0,1] \to G\}_{0 \leq t \leq 1}$ given by

$$u_t(s) := \begin{cases} f\left(\frac{2s}{t+1}\right) & 0 \leq s \leq \frac{1+t}{2}, \\ 1 & \frac{1+t}{2} \leq s \leq 1, \end{cases} \quad v_t(s) := \begin{cases} 1 & 0 \leq s \leq \frac{1-t}{2}, \\ g\left(\frac{2s+t-1}{t+1}\right) & \frac{1-t}{2} \leq s \leq 1. \end{cases}$$

Then the family of maps $\{w_t : [0,1] \to G\}_{0 \leq t \leq 1}$ given by $w_t := u_t \cdot v_t$ yields a homotopy between $f * g$ and $f \cdot g$,

$$w_0 = u_0 \cdot v_0 = (f * 1) \cdot (1 * g) = (f \cdot 1) * (1 \cdot g) = f * g, \quad w_1 = u_1 \cdot v_1 = f \cdot g.$$

We may similarly construct a homotopy between $f * g$ and $g \cdot f$, and it follows that there exists a homotopy between $f \cdot g$ and $g \cdot f$. ☐

Problem 6.

Firstly, there must be some point $x \in M$ with $K(x) > 0$, since M is a compact oriented surface of genus $g \geq 1$. But also (recalling that $\partial M = \varnothing$) we have by Gauss-Bonnet that

$$\iint_M K \, \mathsf{d}A = 2\pi\chi(M) = 2\pi(2 - 2g) \leq 0$$

since $g \geq 1$, and so $K \leq 0$ on some nonempty subset of M. In particular, there's some point $y \in M$ with $K(y) \leq 0$. Therefore, since K is continuous on M, there must be some point $z \in M$ with $K(z) = 0$ by the intermediate value theorem. ☐