Spring 2009 $\boxed{\#1}$

$S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ unit sphere in $\mathbb{R}^3$.

$f: S^2 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$

Claim: f immersion, $f(S^2)$ diffeomorphic to $\mathbb{RP}^2$.

$$df_p = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix} \quad \text{for } p = (x_1, x_2, x_3) \in S^2.$$

$\det(df_p^{\hat{4}}) = x_1(2x_1^2 + 2x_2^2)$

$\det(df_p^{\hat{3}}) = 2x_1(x_1x_2) + 2x_2(x_2^2) = 2x_2(x_1^2 + x_2^2)$

$\det(df_p^{\hat{2}}) = 2x_1(-x_1x_3) + 2x_2(x_2x_3) = 2x_3(x_2^2 - x_1^2)$

$\det(df_p^{\hat{1}}) = x_2(-x_1x_3) - x_1(x_2x_3) = -2x_1x_2x_3$.

where $df_p^{\hat{i}}$ is df_p with the ith row removed.

If $x_1 \neq 0$ then $\det(df_p^{\hat{4}}) \neq 0$.

If $x_2 \neq 0$ then $\det(df_p^{\hat{3}}) \neq 0$.

Otherwise p must be either $(0,0,1)$ or $(0,0,-1)$.

then df_p has rank 2. Thus rank $df_p \geq 2$ $\forall p \in S^2$.

Since S^2 is 2-dim'l, this means f immersion.

Let $[x_1 : x_2 : x_3] \in \mathbb{RP}^2$ be represented by $(x_1, x_2, x_3) \in S^2$.

Then send $[x_1 : x_2 : x_3]$ to $f(x_1, x_2, x_3) \in f(S^2)$.

This is well-defined since $f(-x_1, -x_2, -x_3) = f(x_1, x_2, x_3)$.

We have

$$\begin{tikzcd} S^2 \arrow[rr, "f", "\text{immersion}"'] \arrow[dr, "q"'] & & f(S^2) \subset \mathbb{R}^4 \\ & S^2/\sim = \mathbb{R}P^2 \arrow[ur, "g"'] & \end{tikzcd}$$

Let $(x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3) = (y_1^2 - y_2^2, y_1y_2, y_1y_3, y_2y_3)$

for $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S^2$ to be to points in $f(S^2)$.

If $x_1 = \pm y_1$ then $x_2 = \pm y_2$ and $x_3 = \pm y_3$. The same is true if $x_2 = \pm y_2$ or $x_3 = \pm y_3$.

Otherwise $x_1 \neq \pm y_1$ and $x_2 \neq \pm y_2$ and $x_3 \neq \pm y_3$:

We have $x_1^2 - x_2^2 = y_1^2 - y_2^2$

$$x_1^2 - y_1^2 = x_2^2 - y_2^2 \neq 0$$

Assume WLOG $x_1^2 - y_1^2 = x_2^2 - y_2^2 > 0$

then $\dfrac{x_1^2}{y_1^2} > 1$ and $\dfrac{y_2^2}{x_2^2} < 1$

then $(x_1x_2)^2 \neq (y_1y_2)^2 \Rightarrow x_1x_2 \neq y_1y_2$ a contradiction.

Hence $f(x) = f(y) \Rightarrow x = \pm y \Rightarrow g$ injective.

And g immersion b/c f immersion.

Hence, g diffeomorphism.

Spring 2009 $\boxed{\#2}$

ω closed $\in \Omega^n(\mathbb{R}^{n+1}-\{0\})$

Claim: ω exact $\iff \int_{S^n}\omega = 0$, for S^n unit sphere in $\mathbb{R}^n$.

ω exact $\Rightarrow$ $\omega = d\alpha$ for α in $\Omega^{n-1}(\mathbb{R}^{n+1}-\{0\})$

Then $\int_{S^n}\omega = \int_{S^n} d\alpha = \int_{\partial S^n}\alpha = \int_{\emptyset}\alpha = 0$.

Conversely, ω closed $\Rightarrow$ $\omega \in H^n_{dR}(\mathbb{R}^{n+1}-\{0\})$

and $\omega \in H^n_{dR}(S^n)$. There is an isomorphism

$$H^n_{dR}(S^n) \xrightarrow{\approx} \mathbb{R} \quad \text{sending} \quad \gamma \mapsto \int_{S^n}\gamma$$

Thus if $\int_{S^n}\omega = 0$ then ω must be

the 0 element in $H^n_{dR}(S^n)$ i.e. ω exact.

Spring 2009 $\boxed{\#3}$

$X = e^x \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$ vect. fields on all of $\mathbb{R}^2$

What Z satisfy $[X, Z] = 0$ and $[Y, Z] = 0$.

$$Z = g(x,y)\frac{\partial}{\partial x} + h(x,y)\frac{\partial}{\partial y}$$

$$\begin{aligned}
[X,Z]f &= XZf - ZXf \\
&= X(gf_x + hf_y) - Ze^x f_x \\
&= e^x(g_x f_x + \cancel{g f_{xx}} + h_x f_y + \cancel{h f_{xy}}) - g(e^x f_x + \cancel{e^x f_{xx}}) \\
&\quad - \cancel{h e^x f_{xy}} \\
&= e^x(g_x f_x + h_x f_y - g f_x) = 0
\end{aligned}$$

$$\Rightarrow \quad h_x = 0, \quad g_x = g$$

$$\begin{aligned}
[Y,Z]f &= YZf - ZYf \\
&= Y(gf_x + hf_y) - Zf_y \\
&= g_y f_x + \cancel{g f_{xy}} + h_y f_y + \cancel{h f_{yy}} - \cancel{g f_{xy}} - \cancel{h f_{yy}} \\
&= g_y f_x + h_y f_y = 0
\end{aligned}$$

$$\Rightarrow \quad h_y = 0 \quad g_y = 0.$$

$$\boxed{Z = c_1 e^x \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y}} \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

$$\begin{aligned}
[X,Z]f &= X(c_1 e^x f_x + c_2 f_y) - Z e^x f_x \\
&= e^x\left(\cancel{c_1 e^x f_x} + \cancel{c_1 e^x f_{xx}} + \cancel{c_2 f_{xy}}\right) \\
&\quad - c_1 e^x\left(\cancel{e^x f_x} + \cancel{e^x f_{xx}}\right) - \cancel{c_2 e^x f_{xy}} \\
&= 0 \quad \checkmark
\end{aligned}$$

$[Y,Z]f = 0$ similarly.

Spring 2009 $\boxed{\#4}$

north pole

$$\pi_n(X;x_0) = \{f : S^n \to X \mid f(N) = x_0\}/\simeq$$

where $\simeq$ is basepoint-preserving homotopies.

Compute $\pi_n(T^p)$ for $T^p = \underbrace{S^1 \times \cdots \times S^1}_{p}$.

Hatcher prop 4.1:

A covering space projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ $\forall\, n \geq 2$.

Proof: Let $[f] \in \pi_n(X, x_0)$, $n \geq 2$. Then we have

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

This satisfies the lifting criterion because $\pi_1(S^n) = 0 \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Thus, $\exists\, \tilde{f} : S^n \to \tilde{X}$ a lift $\forall\, [f] \in \pi_n(X, x_0)$

That is, $\forall\, [f] \in \pi_n(X, x_0)$, $\exists\, [\tilde{f}] \in \pi_n(\tilde{X}, \tilde{x}_0)$ s.t. $p_*([\tilde{f}]) = [f]$.

So p_* is surjective, and we know it is injective already.

Claim: $\mathbb{R}^p$ is a covering space of T^p.

Define $p : \mathbb{R}^p \to T^p$ by $p(x_1, \ldots, x_p) = (e^{ix_1}, \ldots, e^{ix_p})$.

This is a covering space because for arbitrary $(e^{i\theta_1}, \ldots, e^{i\theta_p}) \in T^p$, Let $U = \{(e^{i\phi_1}, \ldots, e^{i\phi_p}) \mid \phi_i \in (\theta_i - \varepsilon, \theta_i + \varepsilon)\}$ for small ε. $p^{-1}(U) = \{(x_1, \ldots, x_p) \mid x_i \in (2\pi k + \theta_i - \varepsilon, 2\pi k + \theta_i + \varepsilon), k \in \mathbb{Z}\}$

which is a disjoint collection of sets homeomorphic to U.

Hence. $\pi_n(T^p) \approx \pi_n(\mathbb{R}^p) = 0 \quad \forall\ n \geq 2 \quad \forall\ p.$

And for $n=1$, we know $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ if X, Y path connected. Thus, $\pi_1(T^p) = \mathbb{Z}^p$.

In sum,

$$\boxed{\pi_n(T^p) = \begin{cases} \mathbb{Z}^p & \text{if } n=1 \\ 0 & n \geq 2 \end{cases}}$$

Just use Hatcher 4.2

Spring 2009 $\boxed{\#5}$

Compute $\pi_1(\mathbb{R}^3 - K)$. $K =$ Union of vertical axis $\{x=0, y=0\}$ and unit circle $\{x^2+y^2=1, z=0\}$.

$\mathbb{R}^3 - K$ deformation retracts to $B - K$ where B is the ball of radius 2 centered at the origin.

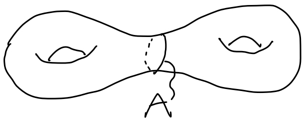
This is homotopy equivalent to the torus, by hollowing out the holes created by K.

Hence $\pi_1(\mathbb{R}^3 - K) \approx \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.

To see the second step, note that the half disc minus a point  is deformation retracts to the circle . Do this homotopy equivalence all around the circle to get $B - K \simeq T^2$.

Spring 2009 $\boxed{\#6}$

$X = M_2$ surface of genus 2. A as below.



Compute $H_n(X, A)$ $\forall\, n \geq 0$.

(X, A) is a good pair since A is a def. retract of a nbd in X.

Then, $H_n(X,A) \approx \tilde{H}_n(X/A)$ and $X/A = T^2 \vee T^2$.

And $\tilde{H}_n(T^2 \vee T^2) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=2 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else} \end{cases}$.

Hence,

$$\boxed{H_n(X, A) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=2 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else} \end{cases}}$$