1. Solve Equations, use $\mathbb{RP}^2 = S^2/\sim$.

2. $I: H^n_{dR}(S^n) \to \mathbb{R}$, $I(\omega) = \int_{S^n} \omega$ is an isomorphism. Stokes, $\partial S^n = \emptyset$.

3. $[A,B] = AB - BA$, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is a basis for vector fields in $\mathbb{R}^2$.

4. If $p: \tilde{X} \to X$ is a covering space, $p^*(\pi_n(\tilde{X}, \tilde{x})) \to \pi_n(X)$ is an isomorphism $\forall$ $n \geq 2$.

5. $\mathbb{R}^3 - K \overset{h.e.}{\cong} T^2$ so $\pi_1(\mathbb{R}^3 - K) = \mathbb{Z} \times \mathbb{Z}$

6. For a good pair (X, A), $H_n(X, A) \simeq \tilde{H}_n(X/A)$ $\forall$ n.

Geometry Topology - Sp '09 :

① $f: S^2 \to \mathbb{R}^4$ given by $f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$
Show f immersion and $f(S^2)$ diffeomorphic to $\mathbb{R}P^2$

Consider the tangent map extended to $\mathbb{R}^3$, i.e. $T_pf: T_p\mathbb{R}^3 \to T_{f(p)}\mathbb{R}^4$ given by the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ where $f_1 = x_1^2 - x_2^2$, $f_2 = x_1x_2$, $f_3 = x_1x_3$, $f_4 = x_2x_3$.
i.e. :

$$T_pf = \left.\begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix}\right|_{p=(x_1,x_2,x_3)}$$

Now, we want to show that $T_pf: T_pS^2 \to T_{f(p)}\mathbb{R}^4$ is injective, so restrict to $T_pS^2 = \{v \in T_p\mathbb{R}^3 : \langle p, v\rangle = 0\}$; now suppose $v \in T_pS^2$ such that $T_pf(v) = 0$. This yields the system (where $v = (a,b,c)$) :

$$\begin{cases} 2ax_1 - 2bx_2 = 0 \\ ax_2 + bx_1 = 0 \\ ax_3 + cx_1 = 0 \\ bx_3 + cx_2 = 0 \end{cases} \Rightarrow \begin{cases} ax_1 = bx_2 \\ ax_2 = -bx_1 \\ ax_3 = -cx_1 \\ bx_3 = -cx_2 \end{cases}$$

$x_1^2 + x_2^2 + x_3^2 = 1$ (since $p \in S^2$)

$ax_1 + bx_2 + cx_3 = 0$ (since $v \in T_pS^2$)

The equation $x_1^2 + x_2^2 + x_3^2 = 1$ implies one of the x_i must be nonzero.

- <u>case $x_1 \neq 0$</u> : $a = \frac{bx_2}{x_1}$ & $b = -\frac{ax_2}{x_1} \Rightarrow a = -a\frac{x_2^2}{x_1^2} = a\left(-\frac{x_2^2}{x_1^2}\right) \Rightarrow \underline{a = 0}$
So now $-bx_1 = 0 \Rightarrow \underline{b = 0}$ since $x_1 \neq 0$. Finally, $-cx_1 = 0 \Rightarrow c = 0$ since $x_1 \neq 0$
So here we have $v = (0,0,0)$.
- <u>case $x_2 \neq 0$</u> : $b = a\frac{x_1}{x_2}$ & $a = -b\frac{x_1}{x_2} \Rightarrow b = -b\frac{x_1^2}{x_2^2} \Rightarrow \underline{b = 0}$; then $ax_2 = 0$ & $-cx_2 = 0 \Rightarrow a = c = 0$ since $x_2 \neq 0 \Rightarrow \underline{v = (0,0,0)}$
- <u>case $x_3 \neq 0$</u> : $b = -\frac{cx_2}{x_3}$ & $a = -\frac{cx_1}{x_3} \Rightarrow -c\frac{x_1^2}{x_3} = -c\frac{x_2^2}{x_3}$
Suppose $c \neq 0$. Then $x_1^2 = x_2^2$. If $x_1 \neq 0$ or $x_2 \neq 0$, we are back to the previous case and get $v = \vec{0}$.

So suppose $x_1^2 = x_2^2 = 0$, hence by $ax_1 + bx_2 + cx_3 = 0$, we get $cx_3 = 0 \Rightarrow c = 0$ since $x_3 \neq 0$, which contradicts our assumption, hence $c = 0$.

With $c = 0$, we get $ax_3 = 0 = bx_3 \Rightarrow a, b = 0$ since $x_3 \neq 0$, hence again $v = (0,0,0)$.

$\rightsquigarrow$ So we've shown $\ker(T_pf : T_pS^2 \to T_{f(p)}\mathbb{R}^4) = 0$, hence $f : S^2 \to \mathbb{R}^4$ is an <u>immersion</u>.

Recall that the projective plane $\mathbb{RP}^2$ is obtained via identifying antipodal points of S^2; we have the standard quotient map: $\pi : S^2 \to \mathbb{RP}^2$ s.t. $\pi(\alpha(p)) = \pi(p)$.

Now note that $\underline{f(\alpha(p))} = f(-x_1, -x_2, -x_3)$

$$= f(x_1^2 - x_2^2, (-x_1)(-x_2), (-x_1)(-x_3), (-x_2)(-x_3))$$

$$= f(x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3) = f(x_1, x_2, x_3) = \underline{f(p)}$$

Hence f also identifies antipodal points.

$\rightarrow$ Therefore, f will be injective on $\mathbb{RP}^2$, and an injective immersion on a compact space is an embedding

<u>Check</u>

② $\omega \in \Omega^n(\mathbb{R}^{n+1}\setminus\{0\})$ closed
Prove ω exact $\Longleftrightarrow \int_{S^n}\omega = 0$?

($\Rightarrow$) Suppose ω exact, i.e. $\exists \alpha \in \Omega^{n-1}(\mathbb{R}^{n+1}\setminus\{0\})$ such that $d\alpha = \omega$.
Then we may apply Stokes:

$$\int_{S^n}\omega = \int_{S^n} d\alpha = \int_{\partial S^n}\alpha = \int_{\emptyset}\alpha = 0$$

($\Leftarrow$) Suppose $\int_{S^n}\omega = 0$

Recall that $S^n \simeq \mathbb{R}^{n+1}\setminus\{0\}$ orientable, connected, compact.

So we get an isomorphism $I: H^n(S^n) \longrightarrow \mathbb{R}$, i.e. $H^n(\mathbb{R}^{n+1}\setminus\{0\}) \cong \mathbb{R}$
$[\omega] \longmapsto \int_{S^n}\omega$

Now, we have $0 = \int_{S^n}\omega = I([\omega]) \Rightarrow [\omega] = I^{-1}(0) = 0$,
hence the class $[\omega]$ is zero, hence ω is exact.

③ Find all vector fields Z on $\mathbb{R}^2$ satisfying $[X,Z] = 0 = [Y,Z]$
where $X = e^x\frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$.

Recall $\left.\begin{aligned} X &= X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} = e^x\frac{\partial}{\partial x} \\ Y &= Y^1\frac{\partial}{\partial x} + Y^2\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\end{aligned}\right\} \Rightarrow \begin{aligned} X^1 &= e^x, \; X^2 = 0 \\ Y^1 &= 0, \; Y^2 = 1\end{aligned}$

and Recall: $[V,W] = \left(V^1\frac{\partial W^1}{\partial x} - W^1\frac{\partial V^1}{\partial x} + V^2\frac{\partial W^1}{\partial y} - W^2\frac{\partial V^1}{\partial y}\right)\frac{\partial}{\partial x}$
$\quad + \left(V^1\frac{\partial W^2}{\partial x} - W^1\frac{\partial V^2}{\partial x} + V^2\frac{\partial W^2}{\partial y} - W^2\frac{\partial V^2}{\partial y}\right)\frac{\partial}{\partial y}$

and then $0 = [X,Z] = \left(e^x\frac{\partial Z^1}{\partial x} - Z^1e^x + 0 - Z^2(0)\right)\frac{\partial}{\partial x}$
$\quad + \left(e^x\frac{\partial Z^2}{\partial x} - Z^1(0) + (0) - Z^2(0)\right)\frac{\partial}{\partial y}$
$\quad = \left(e^x\frac{\partial Z^1}{\partial x} - Z^1e^x\right)\frac{\partial}{\partial x} + \left(e^x\frac{\partial Z^2}{\partial x}\right)\frac{\partial}{\partial y}$

& $0 = [Y,Z] = \left(0 - Z^1(0) + \frac{\partial Z^1}{\partial y} - Z^2(0)\right)\frac{\partial}{\partial x} + \left(0 - Z^1(0) + (1)\frac{\partial Z^2}{\partial y} - Z^2(0)\right)\frac{\partial}{\partial y}$
$\quad = \left(\frac{\partial Z^1}{\partial y}\right)\frac{\partial}{\partial x} + \left(\frac{\partial Z^2}{\partial y}\right)\frac{\partial}{\partial y}$

So then we have

$$\begin{cases} e^x \dfrac{\partial z^1}{\partial x} - z^1 e^x = 0 \Rightarrow e^x \dfrac{\partial z^1}{\partial x} = z^1 e^x \Rightarrow \dfrac{\partial z^1}{\partial x} = z^1 \\ e^x \dfrac{\partial z^2}{\partial x} = 0 \Rightarrow \dfrac{\partial z^2}{\partial x} = 0 \end{cases}$$

Since $e^x \neq 0$ for all x.

$\dfrac{\partial z^1}{\partial y} = \dfrac{\partial z^2}{\partial y} = 0 \Rightarrow$ no occurrence of y

Now, $\dfrac{\partial z^1}{\partial x} = z^1 \Rightarrow z^1 = Ae^x$

and $\dfrac{\partial z^2}{\partial x} = \dfrac{\partial z^2}{\partial y} = 0 \Rightarrow z^2 = k$ constant.

Hence $\boxed{z = Ae^x \dfrac{\partial}{\partial x} + k \dfrac{\partial}{\partial y}}$

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④ Find $\pi_n(T^p)$ for all $n \geq 1$ where $T^p = S^1 \times \cdots \times S^1$ p times.

First, see that $\pi_1(T^p) = \pi_1(S^1 \times \cdots \times S^1) = \pi_1(S^1) \times \cdots \times \pi_1(S^1)$
$= \mathbb{Z} \times \cdots \times \mathbb{Z} = \underline{\mathbb{Z}^p}$

Similarly, we have $\pi_n(T^p) = \pi_n(S^1) \times \cdots \times \pi_n(S^1)$
Recall that $p: \mathbb{R} \to S^1$ is the universal cover
and that for $n \geq 2$, $p_*: \pi_n(\mathbb{R}) \to \pi_n(S^1)$ is an isomorph.
(for covering spaces).

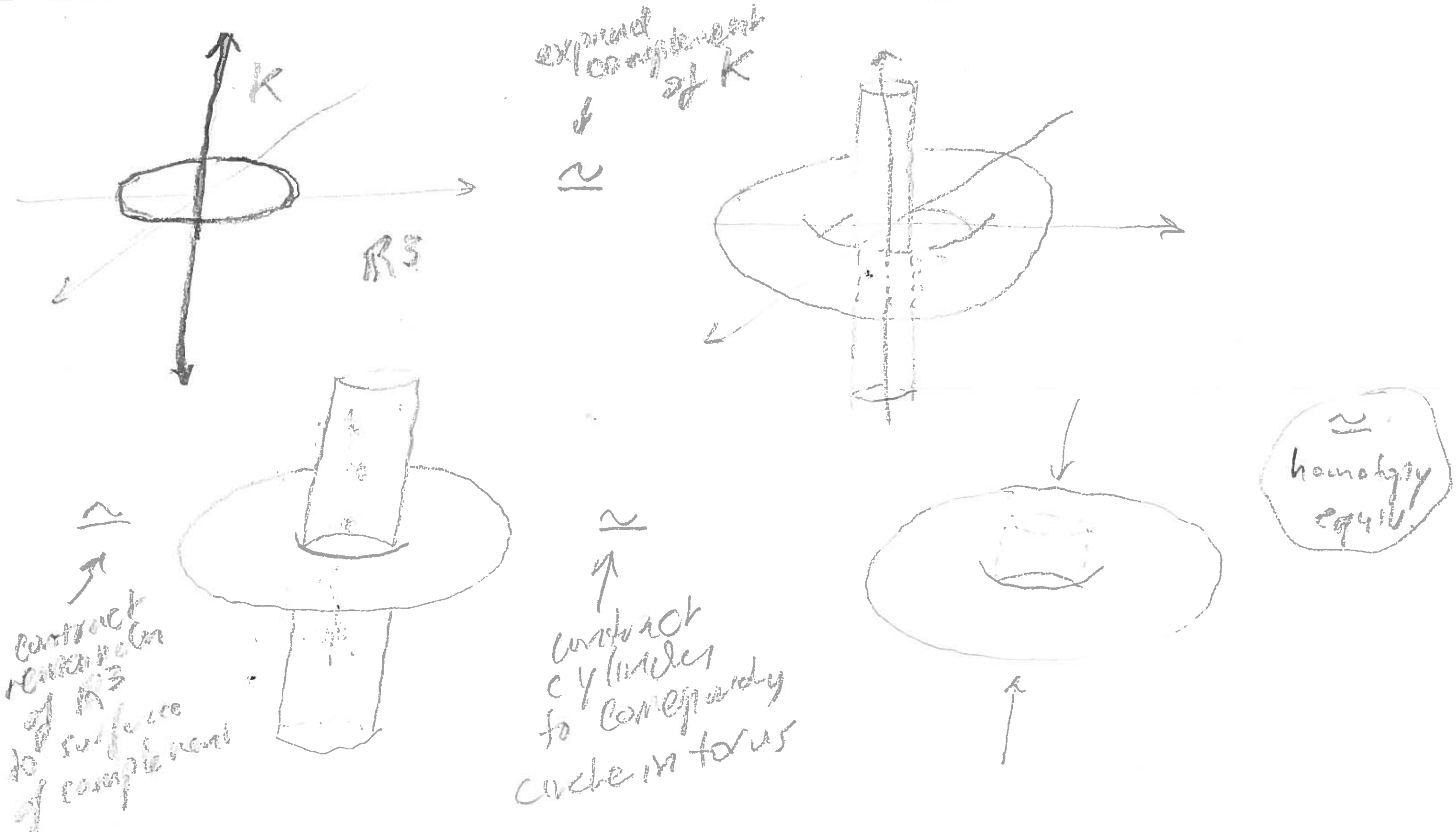
Hence: $\pi_n(S^1) \cong p^*(\pi_n(\mathbb{R})) \cong p^*(0) \cong 0$,

i.e. $\pi_n(S^1)$ trivial, hence $\pi_n(T^p) = 0^p = 0$ trivial as well

$$\Rightarrow \pi_n(T^p) = \begin{cases} \mathbb{Z}^p & n = 1 \\ 0 & n > 1 \end{cases}$$

⑤ Compute $\pi_1(\mathbb{R}^3 \setminus K)$ where $K = \{x=0, y=0\} \cup \{x^2+y^2=1, z=0\}$.

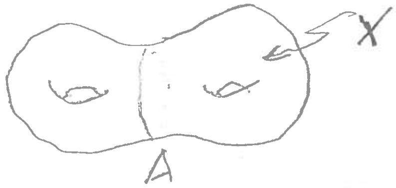
i.e. K = (vertical axis) $\cup$ (unit circle in z-plane)



Hence we see that $\mathbb{R}^3 \setminus K \simeq T^2$

$\Rightarrow \underline{\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}}$.

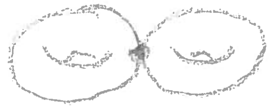
⑥ $X = M_2 =$  and A the curve pictured:



Compute $H_n(X,A)$ for $n \geq 0$

Recall that, for good pair (X,A), we have

$H_n(X,A) \cong \tilde{H}_n(X/A)$.

Now, see that $X/A \cong$  $= T^2 \vee T^2$

Consider the decomposition of $T^2 \vee T^2$ where

$A \simeq T^2$, $B \simeq T^2$, $A \cup B = T^2 \vee T^2$, $A \cap B = pt$.

Then apply Mayer-Vietoris:

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

$$\Rightarrow \cdots \to H_n(pt) \to H_n(T^2) \oplus H_n(T^2) \to H_n(T^2 \vee T^2) \to H_{n-1}(pt) \to \cdots$$

$$\Rightarrow \cdots \to 0 \to H_n(T^2) \oplus H_n(T^2) \to H_n(T^2 \vee T^2) \to 0 \to \cdots$$

$\Rightarrow H_n(T^2) \oplus H_n(T^2) \cong H_n(T^2 \vee T^2)$ by exactness.

Now we know $H_n(T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases} \Rightarrow H_n(T^2 \vee T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z} \oplus \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$

and recall that $H_n(T^2 \vee T^2) \cong \tilde{H}_n(T^2 \vee T^2)$, $n > 0$

$H_0(T^2 \vee T^2) \cong \tilde{H}_0(T^2 \vee T^2) \oplus \mathbb{Z}$, hence:

$$\begin{array}{c} H_n(X,A) \\ \text{\rotatebox{90}{$\cong$}} \end{array}$$

$$\tilde{H}_n(X/A) \cong \tilde{H}_n(T^2 \vee T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z} \oplus \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases} = H_n(X,A)$$