2009, Spring

Problem 1.

Take any $x = (x_1, x_2, x_3) \in \mathsf{S}^2$. Note that at least one of these coordinates is nonzero. We have

$$\mathsf{d}f_x = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix}.$$

If $x_1 \neq 0$, then the last two columns are linearly independent. If $x_2 \neq 0$, then the first and last columns are linearly independent. And if $x_3 \neq 0$, then the first two columns are linearly independent. In any case, $\mathsf{rank}(\mathsf{d}f_x) \geq 2$. So since $2 = \mathsf{dim}_{\mathbb{R}}(\mathsf{T}_x\mathsf{S}^2) = \mathsf{rank}(\mathsf{d}f_x) + \mathsf{dim}_{\mathbb{R}}(\mathsf{ker}(\mathsf{d}f_x))$, we have that $\mathsf{ker}(\mathsf{d}f_x) = 0$, whereby f is an immersion.

Now, it's immediate that $f(x) = f(-x)$ for any $x \in \mathsf{S}^2$, whereby f descends to a well defined (surjective) immersion $\bar{f} : \mathbb{R}\mathsf{P}^2 \to f(\mathsf{S}^2)$, with $\mathbb{R}\mathsf{P}^2$ being the usual quotient $\mathsf{S}^2/\mathbb{Z}_2$. Recall that an embedding is a diffeomorphism onto its image, so we're done if we can show that $\bar{f}$ is an embedding; it remains only to verify that $\bar{f}$ is injective, and this can be (tediously) done directly from the definition of f. □

Problem 2.

Since S^n is a deformation retract of $\mathbb{R}^{n+1} \setminus 0$ via the map $u : \mathbb{R}^{n+1} \twoheadrightarrow \mathsf{S}^n$ given by $u(x) := x/\|x\|$, we have an isomorphism $u^* : \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \to \mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0)$. Moreover we have an isomorphism $I : \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \to \mathbb{R}$ given by $I([\omega]) := \int_{\mathsf{S}^n} \omega$, and so the composite

$$\mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0) \xrightarrow{(u^*)^{-1}} \mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^n) \xrightarrow{I} \mathbb{R}$$

is an isomorphism. A closed n-form $\omega \in \Omega^n(\mathbb{R}^{n+1} \setminus 0)$ is exact if and only if $[\omega] = 0 \in \mathsf{H}^n_{\mathsf{dR}}(\mathbb{R}^{n+1} \setminus 0)$. By the above isomorphism, this is equivalent to $\int_{\mathsf{S}^n} \omega = I \circ (u^*)^{-1}([\omega]) = 0$. □

Problem 3.

If $Z = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ satisfies $[X, Z] = [Y, Z] = 0$, then

$$\begin{aligned}
0 = [X, Z] = XZ - ZX &= e^x \frac{\partial}{\partial x}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) + \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) e^x \frac{\partial}{\partial x} \\
&= e^x \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + e^x f \frac{\partial^2}{\partial x^2} + e^x \frac{\partial g}{\partial x}\frac{\partial}{\partial y} + e^x g \frac{\partial^2}{\partial x \partial y} - f e^x \frac{\partial^2}{\partial x^2} - f e^x \frac{\partial}{\partial x} - g e^x \frac{\partial^2}{\partial x \partial y} - 0 \\
&= e^x \left(\frac{\partial f}{\partial x} - f\right)\frac{\partial}{\partial x} + e^x \frac{\partial g}{\partial x}\frac{\partial}{\partial y}, \\
0 = [Y, Z] = YZ - ZY &= \frac{\partial}{\partial y}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right) - \left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} \\
&= \frac{\partial f}{\partial y}\frac{\partial}{\partial x} + f\frac{\partial^2}{\partial x \partial y} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y} + g\frac{\partial^2}{\partial y^2} - f\frac{\partial^2}{\partial x \partial y} - g\frac{\partial^2}{\partial y^2} \\
&= \frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial}{\partial y}.
\end{aligned}$$

The first equation gives $\frac{\partial f}{\partial x} = f$ and $\frac{\partial g}{\partial x} = 0$, and the second gives $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 0$. Therefore $f = c_1 e^x$ and $g = c_2$ for some constants $c_1, c_2 \in \mathbb{R}$, and $Z = c_1 e^x \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y}$. □

Problem 4.

For path connected spaces X, Y, we have $\pi_j(X \times Y) \cong \pi_j(X) \times \pi_j(Y)$ for all $j \in \mathbb{N}$. Hence

$$\pi_j(\mathsf{T}^p) \cong \prod^p \pi_j(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z}^{\oplus p} & j = 1, \\ 0 & \text{else.} \end{cases}$$

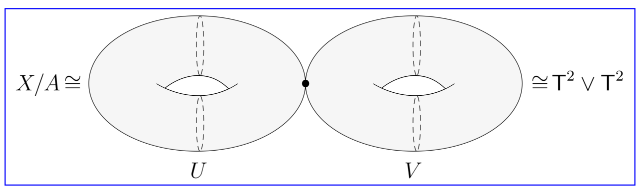
□

Problem 5.

By problem 6 of 2006, Spring, we have $\mathbb{R}^3 \setminus K \cong \mathsf{T}^2$. Then $\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$. □

Problem 6.

Note that (X, A) is a good pair since we can clearly find a neighborhood of A in X which deformation retracts to A, and so $\mathsf{H}_j(X, A) \cong \tilde{\mathsf{H}}_j(X/A)$ for each $j \in \mathbb{Z}$. By collapsing A to a point, we see that $X/A \cong \mathsf{T}^2 \vee \mathsf{T}^2$.



We decompose X/A into two tori U, V, with $U \cap V \cong *$, and by Mayer-Vietoris

$$0 \longrightarrow \mathbb{Z}^{\oplus 2} \longrightarrow \tilde{\mathsf{H}}_2(X/A) \longrightarrow 0 \longrightarrow \mathbb{Z}^{\oplus 4} \xrightarrow{k_1 - \ell_1} \tilde{\mathsf{H}}_1(X/A) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2} \xrightarrow{k_0 - \ell_0} \mathsf{H}_0(X/A) \longrightarrow 0$$

is exact. We compute the (reduced) homologies as follows.

- Immediately, $\tilde{\mathsf{H}}_2(X/A) \cong \mathbb{Z}^{\oplus 2}$.
- By exactness, $\mathsf{ker}(k_1 - \ell_1) \cong 0$. Now note that the map (i_1, j_1) is induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of path connected spaces, and is hence injective. So $\mathsf{im}(\partial_1) \cong \mathsf{ker}(i_0, j_0) \cong 0$, whereby $\mathsf{im}(k_1 - \ell_1) \cong \mathsf{ker}(\partial_1) \cong \tilde{\mathsf{H}}_1(X/A)$. Thus $\tilde{\mathsf{H}}_1(X/A) \cong \mathbb{Z}^{\oplus 4}$.
- Next, $\mathsf{ker}(k_0 - \ell_0) \cong \mathsf{im}(i_0, j_0) \cong \mathbb{Z}$, so by exactness, $\mathsf{H}_0(X/A) \cong \mathsf{im}(k_0 - \ell_0) \cong \mathbb{Z}$.

Hence $\mathsf{H}_j(X, A) \cong \tilde{\mathsf{H}}_j(X/A) \cong \begin{cases} 0 & j = 0, \\ \mathbb{Z}^{\oplus 4} & j = 1, \\ \mathbb{Z}^{\oplus 2} & j = 2, \\ 0 & \text{else.} \end{cases}$ □