Spring 2008 $\boxed{\#1}$

$p: \tilde{X} \to X$ covering. X path connected.

G automorphism group, consisting of homeomorphisms $\varphi: \tilde{X} \to \tilde{X}$

s.t. $p \circ \varphi = p$. $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$, $p(\tilde{x}_0) = x_0$.

For any two $\tilde{x}_0'$, $\tilde{x}_0'' \in p^{-1}(x_0)$, $\exists$ $\varphi \in G$ s.t. $\varphi(\tilde{x}_0'') = \tilde{x}_0'$

Claim: $\exists$ exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}; \tilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \longrightarrow G \longrightarrow 1.$$

Assume $\tilde{X}$ path connected.

Let $[\gamma] \in \pi_1(X; x_0)$. γ lifts uniquely to a path $\tilde{\gamma}$ that starts at $\tilde{x}_0 \in \tilde{X}$. $\tilde{\gamma}$ ends at some $\tilde{x}_0' \in p^{-1}(x_0)$. By hypothesis, $\exists$ homeo $\varphi \in G$ s.t. $\varphi(\tilde{x}_0) = \tilde{x}_0'$. This is unique since deck trasformations are completely determined by where one point is sent.

Thus, we can define a map $f: \pi_1(X; x_0) \longrightarrow G$ that sends $[\gamma] \mapsto \varphi$ as above.

This is a homomorphism: $f([\gamma \cdot \gamma']) = \varphi' \circ \varphi$?

γ lifts to a path from $\tilde{x}_0$ to $\tilde{x}_0'$

γ' lifts to a path from $\tilde{x}_0'$ to $\tilde{x}_0''$

$\Rightarrow$ φ sends $\tilde{x}_0$ to $\tilde{x}_0'$, φ' sends $\tilde{x}_0'$ to $\tilde{x}_0''$.

$\gamma \cdot \gamma'$ lifts to a path from $\tilde{x}_0$ to $\tilde{x}_0''$

and $\varphi' \circ \varphi$ sends $\tilde{x}_0$ to $\tilde{x}_0''$ $\Rightarrow$ $f([\gamma \cdot \gamma']) = \varphi' \circ \varphi$.

What is $Im(f)$?

Let $\varphi \in G$. Then $\varphi(\tilde{x}_0) = \tilde{x}_0'$ for some $\tilde{x}_0' \in p^{-1}(x_0)$.

$\exists$ path from $\tilde{x}_0$ to $\tilde{x}_0'$ which is mapped to a loop γ at x_0 by p. $f([\gamma]) = \varphi$.

So $Im(f) = G = Ker(G \longrightarrow 1)$.

What is $Ker(f)$?

Let $[\gamma] \in \pi_1(X; x_0)$ s.t. $f([\gamma]) = Id$, which sends $\tilde{x}_0$ to $\tilde{x}_0$

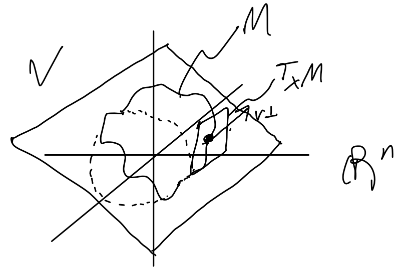
Then γ lifts to a loop $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_0$.

Hence $Ker(f) = p_*(\pi_1(\tilde{X}; x_0)) = im\ p_*$.

Thus the sequence is exact.

Spring 2008 $\boxed{\#2}$

$V \subset \mathbb{R}^n$ vector subspace, $\pi : \mathbb{R}^n \to V$ orthogonal projection.
$M \subset \mathbb{R}^n$ submfd. Claim: $\pi|_M : M \to V$ immersion
$\Longleftrightarrow T_xM \cap V^\perp = \{0\}$ $\forall x \in M$.



$\mathrm{Ker}(d\pi_x) = \{ v \in T_xM : d\pi_x(v) = 0 \}$

WLOG take V to be $\mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n$.

Then $\pi((x_1, \dots, x_n)) = (x_1, \dots, x_m, 0, \dots, 0)$.

and $V^\perp = \{0\}^m \times \mathbb{R}^{n-m}$.

If $T_xM \cap V^\perp = \{0\}$ then $\forall\, v \in T_xM$, $\exists\, v_i \neq 0$ coordinate for $i \in \{1, \dots, m\}$. Then, $d\pi_x(v) \neq 0$.

Hence, $\mathrm{Ker}(d\pi_x) = 0 \Longleftrightarrow d\pi_x$ injective $\forall x \in M$
$\Longleftrightarrow \pi|_M$ immersion.

And $\mathrm{Ker}(d\pi_x) = 0$ $\forall x \in M \Rightarrow d\pi_x(v) \neq 0$ $\forall x \in M$, $\forall v \in T_xM$
$\Rightarrow v \notin V^\perp \Rightarrow T_xM \cap V^\perp = \{0\}$ $\forall x \in M$.

Spring 2008 $\boxed{\#3}$

$f: X \to X \quad f \simeq c_{x_0}$ constant map.

$M_f = X \times [0,1]/\sim \qquad (x,0) \sim (f(x),1)$.

Compute $H_n(M_f;\mathbb{Z})$.

By Hatcher example 2.48

$\exists$ LES:

$$\cdots \to H_n(X) \xrightarrow{\mathbb{1}-f_*} H_n(X) \to H_n(M_f) \to \cdots$$

and since $f \simeq c_{x_0}$, we get

a LES:

$$\cdots \to H_n(X) \xrightarrow{\mathbb{1}-c_{x_0 *}} H_n(X) \to H_n(M_f) \to \cdots$$

For $n>0$, $\mathbb{1} - c_{x_0 *} = \mathbb{1}$

and for $n=0$ $\mathbb{1} - c_{x_0 *} = 0$-map.

Thus, for $n>1$ we 0-maps

going into and out of $H_n(M_f)$

by exactness of isomorphisms on either side.

$\Rightarrow H_n(M_f) = 0$ for $n>1$ by exactness. We have

$$\cdots \to H_1(X) \xrightarrow{\cong} H_1(X) \xrightarrow{0} H_1(M_f) \to H_0(X) \xrightarrow{0} H_0(X) \to H_0(M_f) \xrightarrow{0} 0.$$

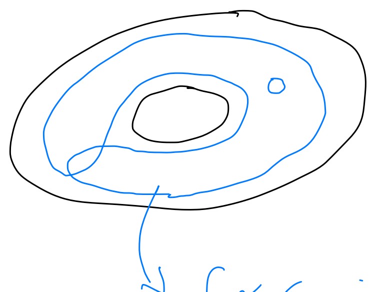
implying $H_1(M_f) \approx H_0(X)$ and $H_0(M_f) \approx H_0(X)$.

Summarizing: $\boxed{H_n(M_f) = \begin{cases} H_0(X) & n=0,1 \\ 0 & \text{else.} \end{cases}}$

?

$f(X)$ contractible in X?

Not necessarily:



$f \simeq c_{x_0}$ in X

$f(X)$ not contractible in X.

Spring 2008 $\boxed{\#4}$

differentiable $f: S^{2n-1} \to S^n$, $n \geq 2$. $\alpha \in \Omega^n(S^n)$.

$\int_{S^n} \alpha = 1$. $f^*(\alpha) \in \Omega^n(S^{2n-1})$.

?

(a) claim: $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

(b) claim: $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ independent of choice of β, α.

independent of β & α?

Recall $H^n(S^n) \xrightarrow{\approx} \mathbb{R}$, $\gamma \mapsto \int_{S^n} \gamma$.

(a)

$H^n_{dR}(S^{2n-1}) = 0 \Rightarrow$ all closed forms are exact.

$d\alpha \in \Omega^{n+1}(S^n) \Rightarrow d\alpha = 0 \Rightarrow d f^*(\alpha) = f^*(d\alpha) = 0$

Thus $f^*(\alpha)$ closed and therefore exact.

That is, $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

(b) $\beta \wedge d\beta \in \Omega^{2n-1}(S^{2n-1}) \Rightarrow \beta \wedge d\beta$ closed, $\in H^{2n-1}_{dR}(S^{2n-1})$

For the same α, pick another $\beta' \in \Omega^{n-1}(S^{2n-1})$

s.t. $f^*(\alpha) = d\beta'$.

Then $\beta \wedge d\beta - \beta' \wedge d\beta' = \beta \wedge d\beta - \beta' \wedge d\beta = (\beta - \beta') \wedge d\beta$.

And $\beta - \beta'$ is exact since $d(\beta - \beta') = d\beta - d\beta' = 0$

and $H^{n-1}_{dR}(S^{2n-1}) = 0$. Hence $\exists\ \gamma \in \Omega^{n-2}(S^{2n-1})$ s.t. $d\gamma = \beta - \beta'$

Then $d(\gamma \wedge d\beta) = d\gamma \wedge d\beta = (\beta - \beta') \wedge d\beta = \beta \wedge d\beta - \beta' \wedge d\beta'$

is exact. This means $[\beta \wedge d\beta] = [\beta' \wedge d\beta'] \in H^{2n-1}_{dR}(S^{2n-1})$

By the isomorphism this means

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta' \wedge d\beta'.$$

Spring 2008 $\boxed{\#5}$

$$\omega = x\,dy\wedge dz + z\,dx\wedge dy + y\,dz\wedge dx \in \Omega^2(S^2)$$

Compute $\int_{S^2}\omega$.

$$d\omega = dx\wedge dy\wedge dz + dz\wedge dx\wedge dy + dy\wedge dz\wedge dx$$
$$= 3\,dx\wedge dy\wedge dz.$$

Stokes $\Rightarrow$ $\int_{S^2}\omega = \int_{\partial D^3}\omega = \int_{D^3} d\omega = \int_{D^3} 3\,dx\wedge dy\wedge dz$

$$= 3\,\mathrm{vol}(D^3) = 3\,\tfrac{4}{3}\pi = \boxed{4\pi}.$$