1. Theorem in Hatcher. Uses lifting criterion and classification of connected covering spaces.
2. Def of immersion, $T_x \pi = \pi \ \forall x \in M$.
3. Mayer Vietoris. Answer $H_i(M_f) = H_i(X) \oplus H_i(S^1)$
4. Solved if n is even
5. Stokes
6. Use x as a coordinate for $\mathbb{R}$ and θ as a coordinate for $\mathbb{RP}^1$.
7. Show hint using maximality of I, then construct an invertible function in I (everywhere positive) using compactness.

Geometry / Topology – Spring '08 :

① $p: \tilde{X} \to X$, X path-connected, $G = G(\tilde{X}) = \{\varphi \in Homeo(\tilde{X}) : p\circ\varphi = \varphi\}$ (= Deck transf. group). Choose $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$ s.t. $p(\tilde{x}_0) = x_0$

Suppose that for any 2 $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$, $\exists \varphi \in G$ s.t. $\varphi(\tilde{x}_0') = \tilde{x}_0$ (i.e. that p is normal covering)

Show that we have short exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow[p_*]{} \pi_1(X, x_0) \longrightarrow G \longrightarrow 1$$

Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$

- Step 1 : We'll show p normal $\Rightarrow H \trianglelefteq \pi_1(X, x_0)$.

Recall that p_* is injective, hence $H \cong \pi_1(\tilde{X}, \tilde{x}_0)$, and recall that changing the basepoint of $\tilde{X}$ from $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ corresponds to conjugating $\pi_1(\tilde{X}, \tilde{x}_0)$ by a path from $\tilde{x}_0$ to $\tilde{x}_1$, i.e. conjugating $H \leq \pi_1(X, x_0)$ by element $\gamma \in \pi_1(X, x_0)$ that lifts to path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$:

$\leadsto \gamma$ loop in X s.t. $\tilde{\gamma}$ path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$, hence for $\alpha \in \pi_1(\tilde{X}, \tilde{x}_0)$, we get $\tilde{\gamma}^{-1}\alpha\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_1)$.

Thus $\gamma \in N_{\pi_1(X,x_0)}(H) \iff \gamma^{-1}H\gamma = H \iff \gamma^{-1}p_*(\pi_1(\tilde{X},\tilde{x}_0))\gamma^{-1} = p_*(\pi_1(\tilde{X},\tilde{x}_0))$

$\iff p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Prop 1.37 Hatcher $\longrightarrow$ $\iff \exists \varphi \in G$ s.t. $\varphi(\tilde{x}_1) = \tilde{x}_0$ $(\tilde{x}_1 \in p^{-1}(x_0))$

Recall that the last condition is satisfied for any pair of points in the fiber of x_0 since p normal, hence

$N_{\pi_1(X,x_0)}(H) = \pi_1(X, x_0) \Rightarrow \underline{H \trianglelefteq \pi_1(X, x_0)}$.

• Step 2: Define the following map:

$$\Phi : \pi_1(X, x_0) = N_{\pi_1(X,x_0)}(H) \longrightarrow G$$

γ with $\tilde{\gamma}$ path b/t $\tilde{x}_0$ & $\tilde{x}_1$ $\longmapsto$ τ sending $\tilde{x}_0$ to $\tilde{x}_1$

This map is surjective since base-point changes in $\tilde{X}$ amongst pts in $p^{-1}(x_0)$ correspond precisely to deck transformations via the classification of covering spaces.

Also, clearly $\ker \Phi = \{\gamma \text{ lifting to loops in } \tilde{X}\}$

$$= p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$$

So now we have the sequence:

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow[p_*]{} \pi_1(X, x_0) \xrightarrow[\Phi]{} G \longrightarrow 1$$

→ p_* injective since p covering map

→ Φ surjective by above

→ $\ker \Phi = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \operatorname{im} p_*$

$\Rightarrow$ seq is short exact

[this problem is proven in Hatcher, Cf. Prop. 1.39]

② $V \subseteq \mathbb{R}^n$ subspace, $\pi: \mathbb{R}^n \to V$ orth. proj w.r.t. $\langle , \rangle$ (i.e. $\ker \pi = V^\perp$). If $M \subseteq \mathbb{R}^n$ submfld, show that $\pi|_M : M \to V$ immersion $\iff T_pM \cap V^\perp = \{0\}$ $\forall p \in M$.

Recall that $\pi: \mathbb{R}^n \to V$ has kernel $V^\perp$ & $\mathbb{R}^n = V \oplus V^\perp$, so we may write $\pi: V \oplus V^\perp \to V$ and we can see that, furthermore, we may write $T_p\pi : V \oplus V^\perp \to V$ since $T_p\mathbb{R}^n \cong \mathbb{R}^n \cong V \oplus V^\perp$ and $T_{\pi(p)} V \cong V$.

Now choose $v \in T_p\mathbb{R}^n \cong V \oplus V^\perp \Rightarrow v = v_1 + v_2$, $v_1 \in V$, $v_2 \in V^\perp$. Then let $\alpha(t)$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$; since $\alpha(t) \in \mathbb{R}^n$, we may also write $\alpha(t) = \alpha_1(t) + \alpha_2(t)$ with $\alpha_1(t) \in V_1$, $\alpha_2(t) \in V^\perp$ and then $\alpha_1'(0) = v_1$, $\alpha_2'(0) = v_2$ in the decomposition above.

So then we see $T_p\pi(v) = (\pi \circ \alpha)'(0) = (\alpha_1)'(0) = v_1$, hence $\ker T_p\pi = V^\perp$, hence $T_p\pi = \pi$ as maps from $\mathbb{R}^n \to V$. So now:

($\Leftarrow$) Suppose $T_pM \cap V^\perp = \{0\}$ for all $p \in M$.

We have just shown that this means $\{0\} = T_pM \cap \ker(T_p\pi)$ for all p, hence $T_p\pi$ restricted to T_pM is injective, hence $\pi|_M : M \to V$ is an immersion.

($\Rightarrow$) Suppose $\pi|_M : M \to V$ is an immersion.

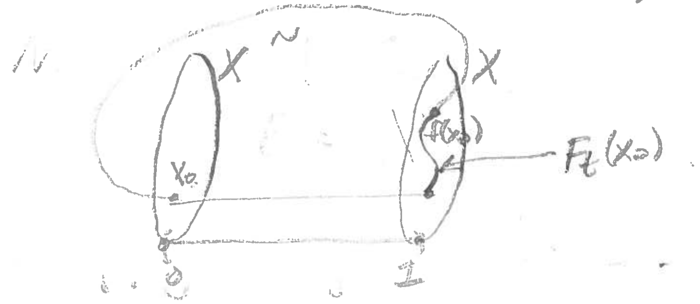
Then $T_p(\pi|_M)$ is an injection for all $p \in M$, hence $\ker T_p\pi \cap T_pM = \{0\}$, but we know $\ker T_p\pi = \ker \pi = V^\perp$, i.e. $V^\perp \cap T_pM = \{0\}$ for all $p \in M$.

(3) $f: X \to X$ homotopic to constant,

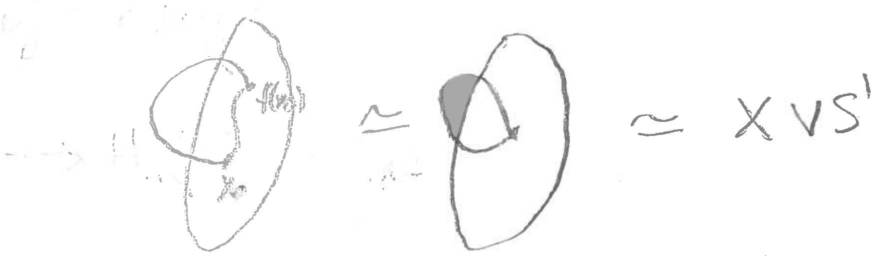
let $M_f = X \times [0,1]/\sim$ where $(x,0) \sim (f(x),1)$.

Compute $H_n(M_f)$ for all n:

Since $f: X \to X$ homotopic to constant, $\exists F_t : X \to X$ with $F_0 = f$, $F_1 \equiv x_0$.
Therefore, for any $x \in X$, we have a path $F_t(x)$ from $F_0(x) = f(x)$ to $F_1(x) = x_0$, hence there exists a path from $(x_0, 1) \to (f(x_0), 1) \sim (x_0, 0)$, i.e. we have a path from $(x_0, 1) \to (x_0, 0)$ in M_f, hence $\{x_0\} \times [0,1] \simeq S^1$:



Now contract $X \times [0,1]$ to X and we have



So then, since each is a good pair w/ base point,

$\tilde{H}_i(M_f) \cong \tilde{H}_i(X \vee S^1) \cong \tilde{H}_i(X) \oplus \tilde{H}_i(S^1)$

Hence $H_i(M_f) \cong H_i(X) \oplus H_i(S^1)$ for $i > 0$

$\tilde{H}_0(M_f) \cong \tilde{H}_0(X) \oplus 0 \Rightarrow H_0(M_f) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$
$\cong \tilde{H}_0(X) \oplus H_0(S^1)$

So $\underline{H_i(M_f) \cong \tilde{H}_i(X) \oplus H_i(S^1)}$

④ $f: S^{2n-1} \to S^n$, $n \geq 2$ differentiable ?

$\alpha \in \Omega^n(S^n)$ such that $\int_{S^n} \alpha = 1$, $f^*(\alpha) \in \Omega^n(S^{2n-1})$ pullback.

(a) Show $\exists \beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$ ($f^*(\alpha)$ exact)

(b) Show that $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β & α:

(a) Step 1: $f^*(\alpha)$ is closed: $df^*(\alpha) = f^*(d\alpha)$, but see that $d\alpha \in \Omega^{n+1}(S^n) = 0$, hence $d\alpha = 0$, hence $f^*(d\alpha) = 0$, hence $df^*(\alpha) = 0$ & $f^*(\alpha)$ is closed

Step 2: Now we may consider the class of $f^*(\alpha)$ in $H^n(S^{2n-1})$, But $H^n(S^{2n-1}) = 0$ since $n \geq 2$, hence $[f^*(\alpha)] = 0$ in $H^n(S^{2n-1})$, hence $f^*(\alpha)$ is exact, i.e. $\exists \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$

(b) Consider $\beta' \in \Omega^{n-1}(S^{2n-1})$ such that $d\beta' = f^*(\alpha)$.

Then $d(\beta - \beta') = d\beta - d\beta' = f^*(\alpha) - f^*(\alpha) = 0$, hence we may consider the class $[\beta - \beta'] \in H^{n-1}(S^{2n-1}) = 0 \Rightarrow [\beta - \beta'] = 0$

$\Rightarrow \beta - \beta'$ exact $\Rightarrow \exists \omega \in \Omega^{n-2}(S^{2n-1})$ such that $d\omega = \beta - \beta'$

$\Rightarrow \beta' = \beta - d\omega$.

Now: $\int_{S^{2n-1}} \beta' \wedge d\beta' = \int_{S^{2n-1}} (\beta - d\omega) \wedge (d\beta + \cancel{d^2\omega}^{\,0})$

$= \int_{S^{2n-1}} (\beta - d\omega) \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta - \int_{S^{2n-1}} d\omega \wedge d\beta$

and see that

$$d(\omega \wedge d\beta) = d\omega \wedge d\beta + \cancel{(-1)^{n-2} \omega \wedge d^2\beta}^{\,0} = d\omega \wedge d\beta$$

and by Stokes: $\int_{S^{2n-1}} d\omega \wedge d\beta = \int_{\partial B^{2n}} d(\omega \wedge d\beta) = \int_{B^{2n}} d^2(\omega \wedge d\beta) = \int_{B^{2n}} 0 = 0$

hence $\int_{S^{2n-1}} \beta' \wedge d\beta' = \int_{S^{2n-1}} \beta \wedge d\beta$

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i.e. $I(f)$ is independent of choice of β.

(5) Let $w \in \Omega^2(S^2)$ be the restriction of
$x\,dy \wedge dz + x\,dz \wedge dy + y\,dz \wedge dx$
to the sphere $S^2 \subseteq \mathbb{R}^3$. Compute $\int_{S^2} w$.

$d(x\,dy \wedge dz + x\,dz \wedge dy + y\,dz \wedge dx)$

$= d(x\,dy \wedge dz) + d(x\,dz \wedge dy) + d(y\,dz \wedge dx)$

$= dx \wedge dy \wedge dz + dx \wedge dz \wedge dy + dy \wedge dz \wedge dx$

$= dx \wedge dy \wedge dz - dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$

$= dx \wedge dy \wedge dz$

now apply Stokes: $\int_{S^2} w = \int_{\partial B^3} w = \int_{B^3} dw = \int_{B^3} dx\,dy \wedge dz$

$= Vol(B^3)$

(6.) Let $f: \mathbb{R} \to \mathbb{RP}^1$ be given by $f(x) = [x, 1]$.

(a) Show $\nexists\ \omega \in \Omega^1(\mathbb{RP}^1)$ such that $f^*(\omega) = p(x)dx$, p polynomial.

(b) Show $\exists\ V \in \mathfrak{X}(\mathbb{RP}^1)$ such that $f^*(V) = p(x)\frac{\partial}{\partial x} \iff \deg p \leq 2$.

(a) Consider the usual atlas on $\mathbb{RP}^1$: $\{(U,u), (V,v)\}$

where $U = \{[x,y] : x \neq 0\}$ and $u([x,y]) = \frac{y}{x}$

Now let $\omega \in \Omega^1(\mathbb{RP}^1)$ and consider in local coordinate u, i.e. $\omega = g(u)\,du$. Then:

$f(x) \in \mathbb{RP}^1$ considered in coordinate u

$$f^*(\omega) = f^*(g(u)du) = g(u(f(x)))\, d(u \circ f(x))$$
$$= g(u([x,1]))\, d(u([x,1]))$$
$$= g\left(\tfrac{1}{x}\right) d\left(\tfrac{1}{x}\right)$$
$$= -g\left(\tfrac{1}{x}\right) \cdot \frac{dx}{x^2}.$$

Now suppose that $g(\frac{1}{x})\frac{1}{x^2} = \sum_{i=0}^{n} a_i x^i$, a polynomial.

$$\Rightarrow \quad g\left(\tfrac{1}{x}\right) = \sum_{i=0}^{n} a_i x^{i+2}$$

$$\Rightarrow \quad g(u) = \sum_{i=0}^{n} a_i \frac{1}{u^{i+2}}$$, i.e. g is not smooth in coordinate u,

which contradicts our hypothesis.

Hence $f^*(\omega)$ cannot be a polynomial.

⑦ M cpt, $C^\infty(M) = \{f : M \to \mathbb{R} \text{ smooth}\}$, $\mathcal{I} \subseteq C^\infty(M)$ max. ideal

Show: $\exists\, x_0 \in M$ s.t. $\mathcal{I} = \{f \in C^\infty(M) : f(x_0) = 0\}$

Suppose this is not true, i.e. for each $x \in M$, $\exists f_x \in \mathcal{I}$ such that $f_x(x) \neq 0$. The f_x are continuous, hence let U_x be the nbhd of x such that $f_x \neq 0$. By construction, we have that $\{U_x\}_{x \in M}$ an open cover of M. M is compact, so take a finite subcover:

$U_{x_1}, \dots, U_{x_k}$ and define $F = f_{x_1}^2 + \dots + f_{x_k}^2 \in \mathcal{I}$

Clearly $F(x) > 0$ for all $x \in \bigcup_{i=1}^{k} U_{x_i} = M$, hence $F > 0$ over all M, hence F is a unit in $C^\infty(M)$.

But, $\mathcal{I}$ is a proper ideal, hence $F \in \mathcal{I}$ is a contradiction since $\mathcal{I}$ may not contain a unit (o/w, $\mathcal{I} = C^\infty(M)$).

Therefore, there must be a point x_0 such that $f(x_0) = 0$ for all $f \in \mathcal{I}$.

Let $\mathcal{I}(x_0) = \{f \in C^\infty(M) : f(x_0) = 0\}$

$\Rightarrow \mathcal{I} \subseteq \mathcal{I}(x_0) \subsetneq C^\infty(M) \Rightarrow \mathcal{I} = \mathcal{I}(x_0)$ by maximality of $\mathcal{I}$.