2008, Spring

Incomplete: 3, 6(b).

Problem 1.

By assumption, G acts transitively on the fiber $p^{-1}(x_0)$, and hence the covering $p : \tilde{X} \twoheadrightarrow X$ is normal. Thus the subgroup $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is normal, and the quotient $\pi_1(X, x_0)/H$ is well defined. This yields the short exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)/H \longrightarrow 1.$$

But also since the covering p is normal, then its group G of deck transformations is isomorphic to $\pi_1(X, x_0)/H$, and this completes the proof. $\square$

Problem 2.

We may write $\mathbb{R}^n \cong V \oplus V^\perp$, with the orthogonal projection $\pi : \mathbb{R}^n \twoheadrightarrow V$ being the identity on the component V and the zero map on the component $V^\perp$. Given $v \in \mathsf{T}_x\mathbb{R}^n \cong \mathbb{R}^n$, by definition of the tangent space $\mathsf{T}_x\mathbb{R}^n$, there's a curve $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Decomposing $v = (v_1, v_2) \in V \oplus V^\perp$ and $\gamma = (\gamma_1, \gamma_2) : (-1, 1) \to V \oplus V^\perp$, we have

$$\mathsf{d}\pi_x(v) = (\pi \circ \gamma)'(0) = \pi(\gamma_1'(0), \gamma_2'(0)) = \pi(v_1, v_2) = v_1 = \pi(v),$$

and so $\mathsf{d}\pi_x = \pi : \mathbb{R}^n \to V$. In particular, this gives $\mathsf{ker}(\mathsf{d}\pi_x) = \mathsf{ker}(\pi) = V^\perp$. Now, $\pi\big|_M : M \to V$ is an immersion if and only if $(\mathsf{d}\pi\big|_M)_x : \mathsf{T}_xM \to \mathsf{T}_{\pi(x)}V$ is injective for every $x \in M$, i.e.

$$0 = \mathsf{ker}((\mathsf{d}\pi\big|_M)_x) = (\mathsf{T}_xM) \cap \mathsf{ker}(\mathsf{d}\pi_x) = (\mathsf{T}_xM) \cap V^\perp$$

for every $x \in M$. $\square$

Problem 4.

(a) We have $\mathsf{d}\alpha = 0$ since $\alpha \in \Omega^n(\mathsf{S}^n)$ is a volume form. Then $\mathsf{d}(f^*\alpha) = f^*\mathsf{d}\alpha = 0$, so the form $f^*\alpha \in \Omega^n(\mathsf{S}^{2n-1})$ is closed. But every closed n-form on S^{2n-1} is exact since $0 < n < 2n-1$ implies $\mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^{2n-1}) \cong 0$. So $f^*\alpha = \mathsf{d}\beta$ for some $\beta \in \Omega^{n-1}(\mathsf{S}^{2n-1})$. $\square$

(b) Let $\beta' \in \Omega^{n-1}(\mathsf{S}^{2n-1})$ also satisfy $f^*\alpha = \mathsf{d}\beta'$. Then $\mathsf{d}(\beta' - \beta) = f^*\alpha - f^*\alpha = 0$, so $\beta' - \beta$ is closed. As above, every closed $(n-1)$-form on S^{2n-1} is exact, and so $\beta' - \beta = \mathsf{d}\gamma$ for some $\gamma \in \Omega^{n-2}(\mathsf{S}^{2n-1})$. Hence

$$\int_{\mathsf{S}^{2n-1}} \beta' \wedge \mathsf{d}\beta' = \int_{\mathsf{S}^{2n-1}} (\beta + \mathsf{d}\gamma) \wedge \mathsf{d}(\beta + \mathsf{d}\gamma) = \int_{\mathsf{S}^{2n-1}} \beta \wedge \mathsf{d}\beta + \int_{\mathsf{S}^{2n-1}} \mathsf{d}\gamma \wedge \mathsf{d}\beta.$$

We're done if we can show that the second integral on the right-hand side vanishes. And indeed,

$$\int_{\mathsf{S}^{2n-1}} \mathsf{d}\gamma \wedge \mathsf{d}\beta = \int_{\mathsf{S}^{2n-1}} \mathsf{d}(\gamma \wedge \mathsf{d}\beta) = \int_{\mathsf{B}^{2n}} \mathsf{d}^2(\gamma \wedge \mathsf{d}\beta) = 0$$

by Stokes. $\square$

Problem 5.

We have $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$, so

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = 3\int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(\mathsf{B}^3) = 4\pi$$

by Stokes. $\square$

Problem 6 (?).

(a) Equip $\mathbb{RP}^1$ with the usual pair of charts $\{(U,t),(V,s)\}$, for

$$U := \{[x:y] \in \mathbb{RP}^1 \mid x \neq 0\}, \quad V := \{[x:y] \in \mathbb{RP}^1 \mid y \neq 0\},$$

and $t : U \to \mathbb{R}, s : V \to \mathbb{R}$ the maps given by $t([x:y]) := y/x$ and $s([x:y]) := x/y$. Now suppose $\omega \in \Omega^1(\mathbb{RP}^1)$ has $f^*\omega = P(x)\mathsf{d}x$. Then on the chart (U,t), we may write $\omega = F(t)\mathsf{d}t$ for some smooth function $F : \mathbb{R} \to \mathbb{R}$, and on this chart

$$P(x)\mathsf{d}x = f^*\omega = f^*(F(t)\mathsf{d}t) = F(t \circ f(x))\mathsf{d}(t \circ f(x)) = F(t([x:1]))\mathsf{d}(t([x:1])) = F\left(\frac{1}{x}\right)\mathsf{d}\left(\frac{1}{x}\right)$$
$$= -\frac{F(1/x)}{x^2}\mathsf{d}x \implies -\frac{F(1/x)}{x^2} = P(x).$$

Writing $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ for some $a_0, \ldots, a_m \in \mathbb{R}$, this gives

$$F\left(\frac{1}{x}\right) = -x^2(a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0) = -a_m x^{m+2} - a_{m-1}x^{m+1} - \cdots - a_0 x^2$$
$$\implies F(t) = -\frac{a_m}{t^{m+2}} - \frac{a_{m-1}}{t^{m+1}} - \cdots - \frac{a_0}{t^2}.$$

At the point $[1:0] \in U$, we have $t([0:1]) = 0/1 = 0$, and $F(0) = \infty$, contradicting F is smooth on the chart (U,t). □

Problem 7.

Background. The manifold M, together with the *sheaf of rings* $\mathscr{C}^\infty_M$, is a *locally ringed space* $(M, \mathscr{C}^\infty_M)$. In this problem we prove that any maximal ideal $\mathscr{I}$ of the ring of *global sections* $\mathscr{C}^\infty(M)$ consists of functions vanishing at some point $x \in M$. The localization of $\mathscr{C}^\infty(M)$ at this point is isomorphic to the *stalk of* $\mathscr{C}^\infty_M$ *at* x, that is, $\mathscr{C}^\infty(M)_{\mathscr{I}} \cong \mathscr{C}^\infty_{M,x}$.

Write $M = \{x_\alpha\}_{\alpha \in A}$ and assume $\mathscr{I}$ isn't of the desired form. Then for every $\alpha \in A$, there's some $f_\alpha \in \mathscr{I}$ with $f_\alpha(x_\alpha) \neq 0$; by continuity, there's some open neighborhood $U_\alpha \subset M$ such that $f_\alpha\big|_{U_\alpha}$ is either strictly positive or strictly negative. By multiplying by the constant function $-1 \in \mathsf{C}^\infty(M)$ if necessary, we may assume w.l.o.g. that $f_\alpha\big|_{U_\alpha} > 0$. Since M is compact, we may choose a finite subcover $\{U_j\}_{j=1}^m$ of the open cover $\{U_\alpha\}_{\alpha \in A}$. Then $f := \sum_{j=1}^m f_j \in \mathscr{I}$ since $\mathscr{I}$ is an ideal, and $f > 0$ on all of M by design. So the function $1/f \in \mathsf{C}^\infty(M)$ is well defined and $f(1/f) = 1$, whereby $\mathscr{I} = (1) = \mathsf{C}^\infty(M)$. But this is impossible since $\mathscr{I} \subsetneq \mathsf{C}^\infty(M)$ by virtue of being a maximal ideal. □