Geo/Top – Spr'07 :

1. Define tangent vector $A \in T_I(M_n(\mathbb{R}))$ by curve $\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R})$, $\alpha(t) = I_n + tA$

(a) For $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$
$$B \longmapsto BX$$

Show R_X is differentiable:

Let $\varphi : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^{n^2}$
$$(a_{ij}) \longmapsto (a_{11}, a_{12}, \dots, a_{nn})$$
; then $(M_n(\mathbb{R}), \varphi)$ is an atlas for $M_n(\mathbb{R})$.

Now we just need that $\varphi \circ R_X \circ \varphi^{-1}$ differentiable:

$$\varphi \circ R_X \circ \varphi^{-1}(a_{11}, a_{12}, \dots, a_{nn}) = \varphi \circ R_X [a_{ij}]$$
$$= \varphi([a_{ij}]X) = \varphi([p_{ij}(a_{11}, \dots, a_{nn})])$$
$$= (p_{11}(a_{11}, \dots, a_{nn}), \dots, p_{nn}(a_{11}, \dots, a_{nn}))$$

where p_{ij} are polynomial the a_{ij} with coefficients the entries of X.

Therefore $\varphi \circ R_X \circ \varphi^{-1}$ has polynomial component functions, hence differentiable in $\mathbb{R}^{n^2}$, hence R_X differentiable.

(b) $A \in T_I(M_n(\mathbb{R}))$, $\xi_A(X) = T_I(R_X)(A)$ vector field.

Compute $[\xi_A, \xi_B]$:

$A \in T_I(M_n(\mathbb{R}))$, $\alpha(t) = I_n + tA$, hence $\alpha(0) = I_n$, $\alpha'(0) = A$;

then $T_I(R_X)(A) = (R_X \circ \alpha)'(0)$:

$$(R_X \circ \alpha)(t) = R_X(\alpha(t)) = R_X(I_n - tA) = X + tAX$$

$\Rightarrow (R_X \circ \alpha)'(t) = AX \Rightarrow T_I(R_X)(A) = (R_X \circ \alpha)'(0) = AX$,

hence $\xi_A(X) = AX$. Then $[\xi_A, \xi_B](X) = \xi_A \circ \xi_B(X) - \xi_B \circ \xi_A(X)$

$= \xi_A(BX) - \xi_B(AX) = ABX - BAX = (AB - BA)X$.

(2.) $C = \{(z,w) \in \mathbb{C}^2 : w^2 = P(z)\} \subseteq \mathbb{C}^2$ where p polynomial of degree 3

(a) If p has no repeated roots, show $C \subseteq \mathbb{C}^2$ submanifold

Consider the map $f : \mathbb{C}^2 \to \mathbb{C}$
$$(z,w) \longmapsto P(z) - w^2$$

Then $f^{-1}(0) = C$, so we may apply the Regular val. thm:
Consider the tangent map:

$$T_pf : T_p\mathbb{C}^2 \to T_{f(p)}\mathbb{C} \quad , \quad p = (z,w) \in f^{-1}(0) = C$$

$$T_pf = [\, P'(z) \;\; -2w \,]$$

Now, since P is over $\mathbb{C}$ (alg. closed) and has no repeated roots, it has the form $P(x) = (x-\alpha)(x-\beta)(x-\gamma)$, i.e. it is a separable polynomial, hence we know that $\gcd(P, P') = 1$, hence P & P' do not share any roots, hence for $p = (z,w) \in f^{-1}(0)$ such that $w^2 = 0$, we have $P'(z) \neq 0$.

On the other hand, if $w^2 \neq 0$, then $-2w \neq 0$.

Hence in either case $T_pf \neq 0$, hence surjective since linear map onto 1-dim'l codomain.

$\longrightarrow$ Thus $f^{-1}(0) = C$ a submfld by reg. val. thm.

(b) If p has no repeated roots, find $\pi_1(C \setminus \{w=0\})$.

3) M smooth mfld; show TM is orientable.

Recall that $TM = \coprod_{p\in M} T_pM = \{(p,v) : p\in M,\ v\in T_pM\}$

Since M a smooth mfld it has an atlas $\{(U,\varphi)\}$ where $\varphi: U \to \mathbb{R}^n$.
Now we can define an atlas $\{(\tilde{U},\tilde{\varphi})\}$ on TM via the following:
Let (U,φ) be a chart on M. Then let.

$$\tilde{U} = \coprod_{p\in U} T_pM \subseteq TM \quad \text{and} \quad \tilde{\varphi}(p,v) = (\varphi(p), T_p\varphi(v)) \in \mathbb{R}^{2n}$$

Now let $(\tilde{U},\tilde{\varphi}), (\tilde{V},\tilde{\psi})$ be charts defined as above, and consider the transition $\tilde{\varphi}\circ\tilde{\psi}^{-1} : \tilde{\psi}(\tilde{U}\cap\tilde{V}) \to \tilde{\varphi}(\tilde{U}\cap\tilde{V})$.

Let $(x_1,\dots,x_{2n}) \in \tilde{\psi}(\tilde{U}\cap\tilde{V}) \subseteq \mathbb{R}^{2n}$, and then:

$$\tilde{\varphi}\circ\tilde{\psi}^{-1}(x_1,\dots,x_{2n}) = \tilde{\varphi}(\psi^{-1}(x_1,\dots,x_n), T_{(x_1,\dots,x_n)}(\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

$$= (\varphi\circ\psi^{-1}(x_1,\dots,x_n), T_{\psi^{-1}(x_1,\dots,x_n)}\varphi\circ T_{(x_1,\dots,x_n)}(\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

Chain rule $\longrightarrow$

$$= (\varphi\circ\psi^{-1}(x_1,\dots,x_n), T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

which is a diffeomorphism since $\varphi\circ\psi^{-1}$ is a diffeomorphism by virtue of it being a transition from a known atlas.

Now to show that $\{(\tilde{U},\tilde{\varphi})\}$ is orientable atlas, we must show that $\det(T_p(\tilde{\varphi}\circ\tilde{\psi}^{-1})) > 0$ for all p; now see:

$$T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})(v_1,\dots,v_{2n})$$

$$= \left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_1,\dots,v_n),\ T_{(x_{n+1},\dots,x_{2n})}\left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})\right)(v_{n+1},\dots,v_{2n})\right)$$

Since $T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})$ is a linear map on $T_{(x_1,\dots,x_n)}\mathbb{R}^n \cong \mathbb{R}^n$, its tangent map induces the same linear map on $T_{(x_{n+1},\dots,x_{2n})}(T_{(x_1,\dots,x_n)}\mathbb{R}^n) \cong \mathbb{R}^n$:

$\rightarrow \left\{ \begin{array}{l} \text{Let } F:\mathbb{R}^m \to \mathbb{R}^n \text{ be linear, } \alpha(t) = p + tv\text{; then:} \\ (F\circ\alpha)(t) = F(\alpha(t)) = F(p+tv) = F(p) + tF(v) \\ \Rightarrow T_pF(v) = (F\circ\alpha)'(0) = F(v) \end{array} \right.$

Therefore,

$$T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})(v_1,\dots,v_{2n})$$
$$= \left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_1,\dots,v_n),\ T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_{n+1},\dots,v_{2n})\right)$$

Now let w_i be the eigenvectors of $T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})$, hence $\{(w_i,0),(0,w_i)\}$ are the eigenvectors of $T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})$.

Now let λ_i be the eigenvalues corresponding to the w_i and see that:

$$T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})(w_i,0) = \left(T_x(\varphi\circ\psi^{-1})(v_i),\ 0\right) = (\lambda_i v_i, 0) = \lambda_i(v_i, 0)$$
$$T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})(0,w_i) = \left(0,\ T_x(\varphi\circ\psi^{-1})(v_i)\right) = (0, \lambda_i v_i) = \lambda_i(0, v_i),$$

hence the eigenvalues of $T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})$ are $\{\lambda_1,\lambda_1,\lambda_2,\lambda_2,\dots\}$, hence

$$\underline{\det(T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})) = \prod_i \lambda_i^2 > 0}$$

since $\lambda_i \neq 0$ for all i since $\varphi\circ\psi^{-1}$ homeomorphism
$\Rightarrow T_x(\varphi\circ\psi^{-1})$ invertible.

(4.) Consider $\omega = dz - y\,dx \in \Omega^1(\mathbb{R}^3)$
Show $f\omega$ is not closed for any nonzero $f: \mathbb{R}^3 \to \mathbb{R}$

First compute the exterior derivative:

$d(f\omega) = d(f\wedge\omega) = df\wedge\omega + f\wedge d\omega$:

- $d\omega = d(dz - y\,dx) = d\cancel{dz}^{=0} - d(y\,dx) = -dy\wedge dx = \underline{dx\wedge dy}$
- $df = f_x\,dx + f_y\,dy + f_z\,dz$
- $df\wedge\omega = (f_x\,dx + f_y\,dy + f_z\,dz)\wedge(dz - y\,dx)$

$$= f_x\,dx\wedge(dz - y\,dx) + f_y\,dy\wedge(dz - y\,dx) + f_z\,dz\wedge(dz - y\,dx)$$

$$= f_x\,dx\wedge dz - f_x\,\cancel{dx\wedge y\,dx}^{0} + f_y\,dy\wedge dz - f_y\,y\,dz\wedge dx + f_z\,\cancel{dz\wedge dz}^{0} - f_z\,dz\wedge y\,dx$$

$$= f_x\,dx\wedge dz + f_y\,dy\wedge dz - f_y\,y\,dy\wedge dx - f_z\,y\,dz\wedge dx$$

$$= f_x\,dx\wedge dz + f_z\,y\,dx\wedge dz + f_y\,dy\wedge dz + f_y\,y\,dx\wedge dy$$

$$= (f_x + f_z y)\,dx\wedge dz + f_y\,y\,dx\wedge dy + f_y\,dy\wedge dz$$

- $f\wedge d\omega = f\,dx\wedge dy$.

So $d(f\omega) = (f_x + f_z y)\,dx\wedge dz + (f_y y)\,dx\wedge dy + (f_y)\,dy\wedge dz + f\,dx\wedge dy$

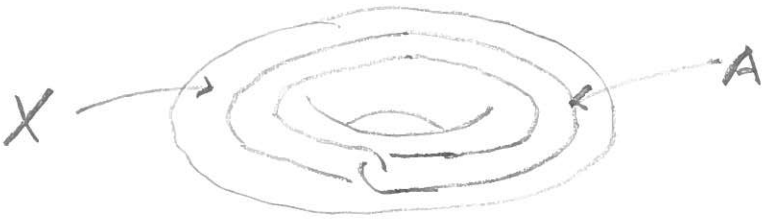
$$= (f_x + f_z y)\,dx\wedge dz + (f_y y + f)\,dx\wedge dy + (f_y)\,dy\wedge dz$$

Suppose now that $f\omega$ is closed, i.e. $d(f\omega) = 0$.

Then: $\left.\begin{array}{l} f_x + f_z y = 0 \\ f_y y + f = 0 \\ f_y = 0 \end{array}\right\} \Rightarrow (0)y + f = 0 \Rightarrow f = 0$, a contradiction since $f \neq 0$ by hypothesis.

<u>So $f\omega$ not closed</u>

(5.) A the pictured knot, $X = S^1 \times D^2$ solid torus.
Show $\nexists$ deformation retract of X onto A:



Recall that if A is a deformation retract of X, then the induced homomorphism on fund. grp of the inclusion $i : A \hookrightarrow X$ is an isomorphism

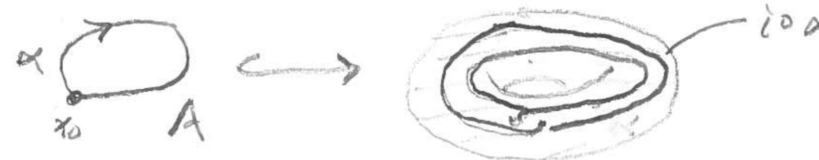
$$i_* : \pi_1(A) \xrightarrow{\cong} \pi_1(X).$$

So suppose that A def. retract of X. Then

$$i_* : \pi_1(A) \to \pi_1(X) \text{ isomorphism}$$

But now let the loop α be a generator for $\pi_1(A, x_0) \cong \pi_1(S^1, x_0) \cong \mathbb{Z}\langle\alpha\rangle$. Then:

$$i_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$$



and see that $i \circ \alpha$ is a nullhomotopic path in $X = S^1 \times D^2$ since solid, i.e. $i_*(\alpha) = 1$,
hence i_* must be trivial since α was the generator of $\pi_1(A)$, hence i_* not an isomorphism, a contradiction; hence, A <u>not def. ret. of X</u>.

6.) Construct top. space X s.t. $H_0(X)=\mathbb{Z}$, $H_3(X)=\mathbb{Z}/5\mathbb{Z}$, $H_5(X)=\mathbb{Z}$, and all others trivial

Our space will consist of 1 0-cell v, 1 3-cell α, 1 4-cell β, and 1 5-cell γ, with the following chain complex:

$$0 \to \mathbb{Z}\{\gamma\} \xrightarrow{\partial_5} \mathbb{Z}\{\beta\} \xrightarrow{\partial_4} \mathbb{Z}\{\alpha\} \xrightarrow{\partial_3} 0 \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{Z}\{v\} \xrightarrow{\partial_0} 0$$

$$\gamma \longmapsto 0$$

$$\beta \longmapsto 5\alpha$$

So we obtain our space by starting with a 0-cell v.
Now attach α to v along its boundary.
Then attach β to α by wrapping the boundary of β around α 5 times.
Finally, attach γ so that β is the boundary of γ, i.e. $\partial e^5_\gamma = e^4_\beta$ (i.e. identify the boundary of γ with β).

(<u>check</u>:

$$H_3(X) = \ker\partial_3/\operatorname{im}\partial_4 = \mathbb{Z}\{\alpha\}/\mathbb{Z}\{5\alpha\} \cong \mathbb{Z}/5\mathbb{Z}$$

$$H_4(X) = \ker\partial_4/\operatorname{im}\partial_5 = 0/0 = 0$$

$$H_5(X) = \ker\partial_5/\operatorname{im}\partial_6 = \ker\partial_5 = \mathbb{Z}\{\gamma\} \cong \mathbb{Z}$$)