1. Regular Value Theorem

2. a) Cohomology groups of the cochain complex $0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \to \dots$

b) Given any smooth f, $g = \int_0^x f(t)\,dt$ satisfies $d_0 g = f\,dx$, so $H^0_{dR}(\mathbb{R}) = \mathbb{R}$, $H^1_{dR}(\mathbb{R}) = 0$.

3. $X \overset{h.e.}{\simeq} S^n \vee S^1 \vee S^1$

4. Stokes

$$\begin{array}{ccc} H_1^{dR}(T) \otimes H_1^{dR}(T) & \xrightarrow{h^* \otimes h^*} & H_1^{dR}(S) \otimes H_1^{dR}(S) \\ \downarrow \wedge & & \\ H_2^{dR}(T) & & \end{array}$$

Write down a basis for $H_1^{dR}(T) \otimes H_1^{dR}(T)$.

5. A dimension argument

6. Think of S^3 as $\mathbb{R}^3 - \{\infty\}$, let one circle pass through ∞ (z-axis). So $U \overset{h.e.}{\simeq}$ Torus.
Use "Inverse" Mayer Vietoris on H.

7. Covering Spaces and Van Kampen

(1.) $M = \{(x,y,z,w) : x^2 + xy^3 + yz^4 - w^5 = -1\} \subseteq \mathbb{R}^4$ smooth mfld?

Regular value theorem: Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$(x,y,z,w) \longmapsto x^2 + xy^3 + yz^4 - w^5 + 1$.

Now we have $M = f^{-1}(0)$, hence must only show 0 reg. value.

Choose $p \in f^{-1}(0)$;

$$T_p f : T_p\mathbb{R}^4 \longrightarrow T_0\mathbb{R} \cong \mathbb{R}$$

$$T_p f = \left[\, 2x + y^3 \quad 3xy^2 + z^4 \quad 4yz^3 \quad -5w^4 \,\right]\Big|_{p=(x,y,z,w)}$$

This is a linear map into $\mathbb{R}$, hence to be surjective we only need that it be nonzero, i.e. one entry of $T_p f$ must be nonzero.

Recall that $p = (x,y,z,w) \in f^{-1}(0)$, hence $x^2 + xy^3 + yz^4 - w^5 = -1$, hence not all of x,y,z,w may be zero.

Case $x \neq 0$: Then $2x + y^3 \neq 0$ (done) or $y \neq 0$. If $y \neq 0$, then $3xy^2 + z^4 \neq 0$ (done) or $z \neq 0$. If $z \neq 0$, then $4yz^3 \neq 0$ (done)

Case $y \neq 0$: Then $2x + y^3 \neq 0$ (done) or $x \neq 0$. If $x \neq 0$, then $3xy^2 + z^4 \neq 0$ (done) or $z \neq 0$. If $z \neq 0$ then $4yz^3 \neq 0$ (done).

Case $z \neq 0$: then $4yz^3 \neq 0$ (done) or $y = 0$. If $y = 0$ then $3xy^2 + z^4 = z^4 \neq 0$ (done).

Case $w \neq 0$: $-5w^4 \neq 0$ (done).

Thus $T_p f$ is surjective, hence $f^{-1}(0) = M$ is a submanifold.

② (a) Def'n of $H^i_{dR}(M)$
(b) Compute de Rham cohomology of $\mathbb{R}$ from def'n.

(a) $H^i_{dR}(M)$ are the cohomology groups of the cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots$$

(b) Consider the cochain complex:

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d_0} \Omega^1(\mathbb{R}) \xrightarrow{d_1} 0$$

Then we have $H^1(\mathbb{R}) := \ker d_1 / \operatorname{im} d_0$ & $H^0(\mathbb{R}) := \ker d_0$

A form in $\Omega^1(\mathbb{R})$ is of the form $f\,dx$ where $f \in C^\infty(\mathbb{R})$
See that we have $g(x) = \int_0^x f(t)\,dt \in \Omega^0(\mathbb{R})$ such that $d_0 g = f\,dx$ by the Fundamental Thm of Calculus, hence d_0 is surjective, i.e. $\operatorname{im} d_0 = \Omega^1(\mathbb{R})$.

On the other hand, $\ker d_0 = \{ f \in \Omega^0(\mathbb{R}) : d_0 f = 0 \}$
$= \{ \text{Constant functions on } \mathbb{R} \}$
$\cong \mathbb{R}$

So now: $H^1(\mathbb{R}) = \Omega^1(\mathbb{R}) / \Omega^1(\mathbb{R}) \cong 0$

$H^0(\mathbb{R}) = \mathbb{R}$

(3.) $X := S^n/_{\{q,r,s\}}$ = S^n with 3 pts identified

Let $p = [q] \in X$ and compute $\pi_1(X,p)$

$X =$ 

$\simeq$ 

$\simeq S^n \vee S^1 \vee S^1$

by Seifert-van Kampen

Hence $\pi_1(X,p) \cong \pi_1(S^n \vee S^1 \vee S^1, p) \cong \pi_1(S^n) * \pi_1(S^1) * \pi_1(S^1)$

$$= \begin{cases} \mathbb{Z}*\mathbb{Z}*\mathbb{Z} & n=1 \\ \mathbb{Z}*\mathbb{Z} & n>1 \end{cases}$$

(4.) $S^3 = \{(x,y,z,w) : x^2+y^2+z^2+w^2 = 1\} \subset \mathbb{R}^4$

Let $\omega = w\,dx\wedge dy\wedge dz$ and compute $\int_{S^3}\omega$.

See that $d\omega = d(w\,dx\wedge dy\wedge dz) = dw\wedge dx\wedge dy\wedge dz$ and apply Stokes.

$$\int_{S^3}\omega = \int_{\partial B^4}\omega = \int_{B^4} d\omega = \int_{B^4} dw\wedge dx\wedge dy\wedge dz = \underline{\mathrm{vol}(B^4)}$$

(5.) Σ closed, orientable surface

$g(\Sigma) = \frac{1}{2} \dim_{\mathbb{R}} H^1_{dR}(\Sigma)$ (genus)

Let S, T be closed, ornt surfaces of genus $g(S), g(T)$ s.t. $g(S) < g(T)$ show that for smooth map $h: S \to T$, $\deg h = 0$.

(Given: for closed, ornt Σ, wedge yields skew-symm bilinear function $H^1(\Sigma) \otimes H^1(\Sigma) \to H^2(\Sigma) \cong \mathbb{R}$)

Recall S, T surfaces, hence degree given by the map

$$h^*: H^2(T) \to H^2(S)$$
$$[\alpha] \mapsto \deg h\,[\alpha]$$

Now consider the following diagram:

$$\begin{tikzcd}
H^1(T) \otimes H^1(T) \arrow[r, "h^* \otimes h^*"] \arrow[d, "\wedge"'] & H^1(S) \otimes H^1(S) \arrow[d, "\wedge"] \\
H^2(T) \arrow[r, "h^*"] & H^2(S) \\
{[\alpha]} \arrow[r, mapsto] & (\deg h)[\alpha]
\end{tikzcd}$$

Recall that $g(S) < g(T)$, hence:

$$\dim H^1(S) = 2g(S) < 2g(T) = \dim H^1(T),$$

hence $\dim(H^1(T) \otimes H^1(T)) > \dim(H^1(S) \otimes H^1(S))$, hence the map $h^* \otimes h^*$ must have non-trivial kernel by rank-nullity; let $\alpha \otimes \beta \in \ker(h^* \otimes h^*)$.

Then $\alpha \wedge \beta \in H^2(T)$ and:

$$h^*(\alpha \wedge \beta) = h^*\alpha \wedge h^*\beta = \wedge\left(h^* \otimes h^*(\alpha \otimes \beta)\right)$$
$$= \wedge(0) = 0$$

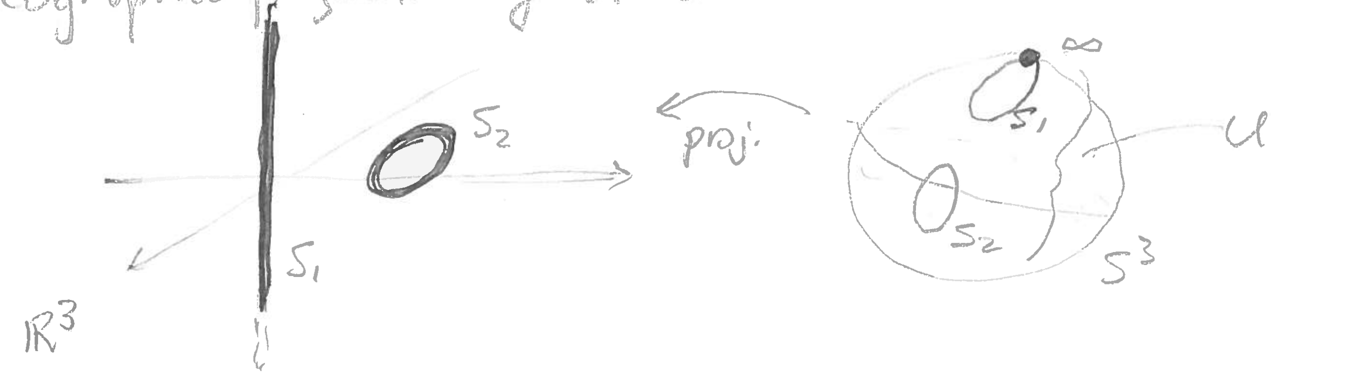
hence $\ker h^* \neq 0$, hence $\underline{\deg h = 0}$

(6.) $U =$  $\subseteq S^3$ (drawn in $\mathbb{R}^3 = S^3 \setminus \{\infty\}$ here)

$H =$  $\subseteq S^3$

Find $H_i(S^3 \setminus U)$ and $H_i(S^3 \setminus H)$:

unlink: Consider S^3 as $\mathbb{R}^3 \cup \{\infty\}$ and translate one of the circles in U so that it passes through infinity; then the stereographic projection of $U \subseteq S^3$ onto $\mathbb{R}^3$ looks like:



Therefore, we now have $S^3 \setminus U \subseteq S^3 \setminus \{\infty\} \cong \mathbb{R}^3$ viewable as $\mathbb{R}^3 - \{\text{circle \& axis}\}$.

Let $A = S^3 \setminus S_1 \simeq S^3 \setminus S^1 \simeq S^1$

$B = S^3 \setminus S_2 \simeq S^3 \setminus S^1 \simeq S^1$

and then $A \cap B = S^3 \setminus (S_1 \cup S_2) = S^3 \setminus U$ & $A \cup B = S^3$.

Hence: $H_i(A) = H_i(B) = H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}$

$H_i(A \cup B) = H_i(S^3) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ 0 & i = 1, 2, i > 3 \end{cases}$

Now apply Mayer-Vietoris:

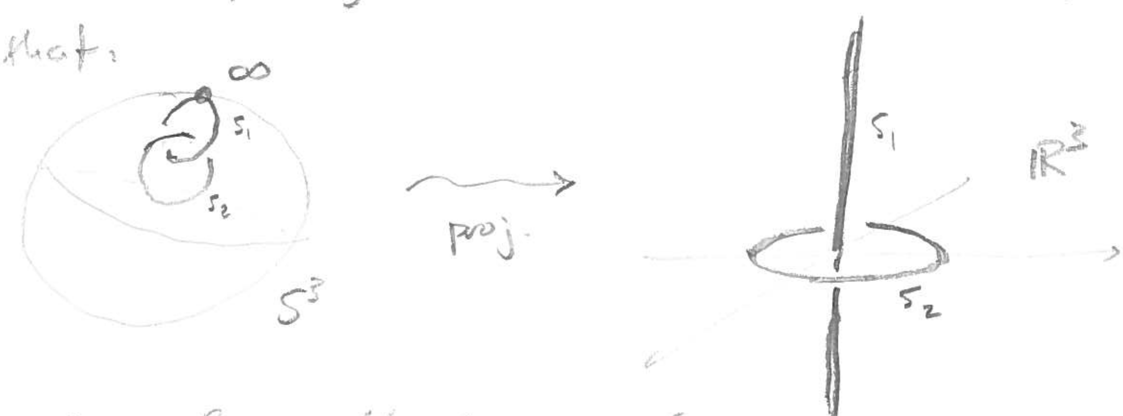
$\to H_3(A \cap B) \to H_3(A) \oplus H_3(B) \to H_3(A \cup B) \to H_2(A \cap B) \to H_2(A) \oplus H_2(B)$
$\to H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(A \cup B) \to H_0(A \cap B)$
$\to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$

$\Rightarrow 0 \to H_3(S^3 \setminus U) \to 0 \to \mathbb{Z} \to H_2(S^3 \setminus U) \to 0 \to 0 \to H_1(S^3 \setminus U) \to \mathbb{Z} \oplus \mathbb{Z}$
$\to 0 \to H_0(S^3 \setminus U) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$.

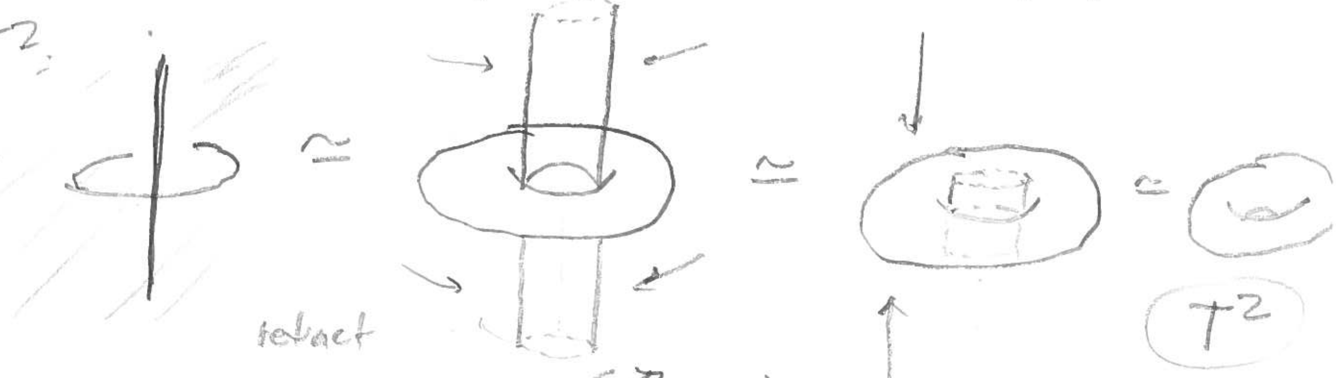
Clearly $H_3(S^3 \setminus U) \cong 0$, $H_2(S^3 \setminus U) \cong \mathbb{Z}$, $H_1(S^3 \setminus U) \cong \mathbb{Z} \oplus \mathbb{Z}$ by exactness, and $H_0(S^3 \setminus U) \cong \mathbb{Z}$ by path-connectedness.

So we have: $H_i(S^3 \setminus U) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$

<u>Hopf link</u>: Again, translate one of the circles so that it passes thru infinity in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Then project and see that:



Now its clear from this projection that $S^3 \setminus H \subseteq S^3 \setminus \{\infty\} \cong \mathbb{R}^3$, hence we may view the projection as $S^3 \setminus H$ itself, and see that the projection is $\simeq T^2$:



Hence $H_i(S^3 \setminus H) \cong H_i(T^2) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$

(7.) X = bouquet of $n+1$ circles $= \bigvee^{n+1} S^1$

F_n = free group on n generators

(a) Prove that $\pi_1(X, p) \cong F_{n+1}$

(b) $H \leq F_{n+1}$, $[F_{n+1} : H] = k$. Show that $H = F_{kn+1}$

(a) Let $X_2 = S^1 \vee S^1$. Then let $A =$  $\simeq S^1$

$B =$  $\simeq S^1$

Then $A \cap B = x \simeq pt.$ and $A \cup B = X_2$

Now apply Seifert-van Kampen; note that both $i_1^*: \pi_1(A \cap B) \to \pi_1(A)$ and $i_2^*: \pi_1(A \cap B) \to \pi_1(B)$ are trivial since $\pi_1(A \cap B) = \pi_1(pt)$ is trivial. Hence:

$$\pi_1(X_2) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1(\omega)\, i_2(\omega)^{-1} \rangle$$

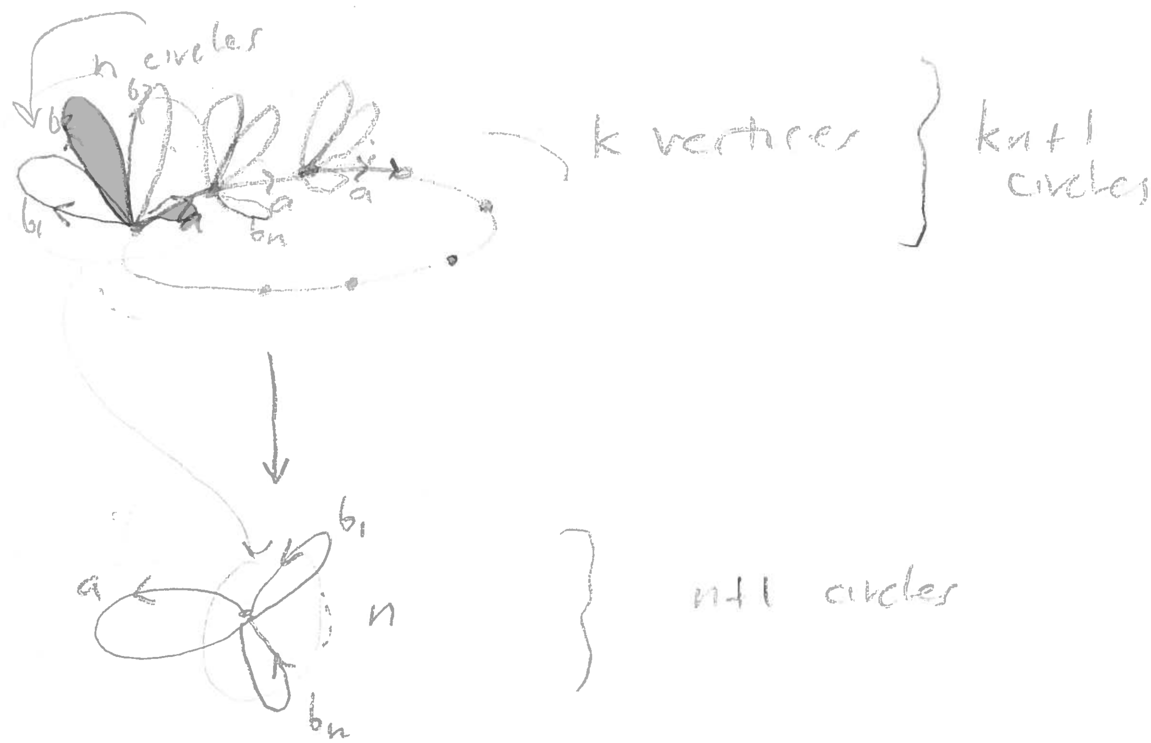
$$\cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} = F_2$$

Now let $X_n = \bigvee^n S^1$ and suppose for induction that $\pi_1(X_n) = F_n$, and let $A \cong X_n$ and $B \simeq S^1$ so that $A \cup B = X_{n+1}$ and $A \cap B \simeq pt.$ Both inclusions again have trivial induced homomorphisms since $\pi_1(A \cap B) = \pi_1(pt)$ is trivial, hence

$$\pi_1(X_{n+1}) = \pi_1(A) * \pi_1(B) = \pi_1(X_n) * \pi_1(S^1) = F_n * \mathbb{Z} = \underline{\underline{F_{n+1}}}$$

(b): Let $H \leq F_{n+1}$ with $[F_{n+1} : H] = k$, i.e. $H \leq \pi_1(V^{n+1}S^1)$,
hence let $p: \tilde{X} \to V^{n+1}S^1$ be the covering space corresponding to H, i.e. the covering space with $\pi_1(\tilde{X}) \cong p_*(\pi_1(\tilde{X})) = H$.
By the above, $\tilde{X}$ will be a k-sheeted covering space of $V^{n+1}(S^1)$; now we must only show that such a cover is a wedge of $kn+1$ copies of S^1:

See the following; consider a circle S^1 with k vertices chosen; and attach n circles at each vertex



This is the k-sheeted covering of $V^{n+1}S^1$,
hence $\pi_1(\tilde{X}) = F_{kn+1}$, i.e. $H = F_{kn+1}$