2006, Spring

Problem 1.

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by $f(x, y, z, w) := x^2 + xy^3 + yz^4 - w^5 + 1$. To show that $X := f^{-1}(0) \subset \mathbb{R}^4$ is a manifold, it's enough to show that the linear map

$$\mathsf{d}f_{(x,y,z,w)} = \begin{pmatrix} 2x + y^3 & 3xy^2 + z^4 & 4yz^3 & -5w^4 \end{pmatrix}$$

from X to $\mathbb{R}$ is surjective for all $(x, y, z, w) \in X$; so we need at least one entry in this matrix to be nonzero. To see this, let $(x, y, z, w) \in X$ and observe that at least one coordinate is nonzero by definition of X.

- Say $x \neq 0$. If $y = 0$ then $2x + y^3 \neq 0$. If $y \neq 0$ and $z = 0$ then $3xy^2 + z^4 \neq 0$. If $y, z \neq 0$ then $4yz^3 \neq 0$.
- Say $y \neq 0$. If $z \neq 0$ then $4yz^3 \neq 0$. If $z = 0$ and $x \neq 0$ then $3xy^2 + z^4 \neq 0$. If $z, x = 0$ then $2x + y^3 \neq 0$.
- Say $z \neq 0$. If $y \neq 0$ then $4yz^3 \neq 0$. If $y = 0$ then $3xy^2 + z^4 \neq 0$.
- Say $w \neq 0$. Then $-5w^4 \neq 0$.

□

Problem 2.

(a) Given a manifold X, the *de Rham cochain complex* $(\Omega^\bullet(X), \mathsf{d}^\bullet)$ is defined in each degree $j \in \mathbb{Z}$ by $\Omega^j(X) := \{\omega \text{ a smooth } j\text{-form on } X\}$ and $\mathsf{d}^j : \Omega^j(X) \to \Omega^{j+1}(X)$ the usual exterior differential. The *j-th de Rham cohomology group* of X is the quotient $\mathsf{H}^j_{\mathsf{dR}}(X) := \mathsf{ker}(\mathsf{d}^j)/\mathsf{im}(\mathsf{d}^{j-1})$.

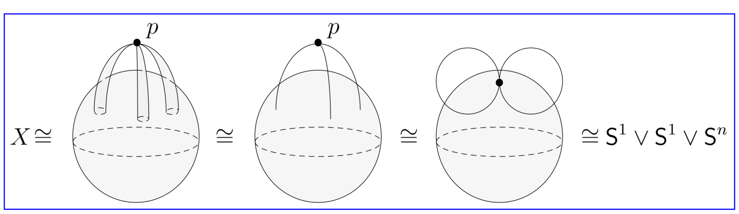
(b) Firstly, $\mathsf{H}^j_{\mathsf{dR}}(\mathbb{R}) \cong 0$ for any $j \geq 2$ since $\Omega^j(\mathbb{R}) = 0$ in this case. Now note that both $\Omega^0(\mathbb{R})$ and $\Omega^1(\mathbb{R})$ are canonically isomorphic to $\mathsf{C}^\infty(\mathbb{R})$. Then

$$\mathsf{H}^0_{\mathsf{dR}}(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^0) \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid \mathsf{d}f = 0\} \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid f \text{ a constant}\} \cong \mathbb{R}.$$

Moreover, any $f \in \Omega^1(\mathbb{R})$ may be written as $f = \mathsf{d}g$ for $g \in \Omega^0(\mathbb{R})$ given by $g(x) := \int_{-\infty}^x f(x)\mathsf{d}x$, and so $\mathsf{im}(\mathsf{d}^0) = \Omega^1(\mathbb{R})$. Thus $\mathsf{H}^1_{\mathsf{dR}}(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^1)/\mathsf{im}(\mathsf{d}^0) \cong \Omega^1(\mathbb{R})/\Omega^1(\mathbb{R}) \cong 0$. □

Problem 3.

By pinching the points q, r, s together and then transforming the shape as shown, we obtain a wedge of S^n with two copies of S^1.



Hence by van Kampen, $\pi_1(X) \cong \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^n) \cong \begin{cases} \mathsf{F}_3 & n = 1, \\ \mathsf{F}_2 & n \geq 2, \end{cases}$. □

Problem 4.

The canonical volume form on $\mathbb{R}^4$ with coordinates (x,y,z,w) is $\mathsf{d}x\wedge\mathsf{d}y\wedge\mathsf{d}z\wedge\mathsf{d}w$. Hence

$$\int_{\mathsf{S}^3}\omega=\int_{\mathsf{B}^4}\mathsf{d}\omega=\int_{\mathsf{B}^4}\mathsf{d}w\wedge\mathsf{d}x\wedge\mathsf{d}y\wedge\mathsf{d}z=-\int_{\mathsf{B}^4}\mathsf{d}x\wedge\mathsf{d}y\wedge\mathsf{d}z\wedge\mathsf{d}w=-\mathsf{vol}(\mathsf{B}^4)$$

by Stokes. □

Problem 5.

Background. The pairing $\smile: \mathsf{H}^1_{\mathsf{dR}}(T)\otimes\mathsf{H}^1_{\mathsf{dR}}(T)\to\mathbb{R}$ referenced in this problem is the *cup product* discussed in problem 7 of 2005, Fall.

Since $\frac{1}{2}\mathsf{dim}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(S))=\mathsf{g}(S)<\mathsf{g}(T)=\frac{1}{2}\mathsf{dim}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(T))$, the map $h^*:\mathsf{H}^1_{\mathsf{dR}}(T)\to\mathsf{H}^1_{\mathsf{dR}}(S)$ is has nontrivial kernel. So, suppose $\alpha\in\mathsf{ker}(h^*)$ is nonzero. Then the map

$$\alpha\smile(\cdot):\mathsf{H}^1_{\mathsf{dR}}(T)\to\mathsf{Hom}_{\mathbb{R}}(\mathsf{H}^1_{\mathsf{dR}}(T),\mathbb{R})$$

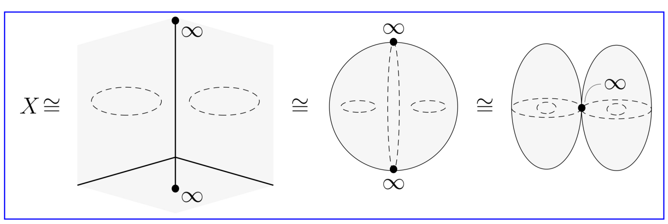
given by $\eta\mapsto\alpha\smile\eta:=\int_T\alpha\wedge\eta$ is nonzero, since the pairing $\smile: \mathsf{H}^1_{\mathsf{dR}}(T)\otimes\mathsf{H}^1_{\mathsf{dR}}(T)\to\mathbb{R}$ given by $\omega\smile\eta:=\int_T\omega\wedge\eta$ is nondegenerate. Thus there's some element $\beta\in\mathsf{H}^1_{\mathsf{dR}}(T)$ such that $\int_T\alpha\wedge\beta\neq0$, and so

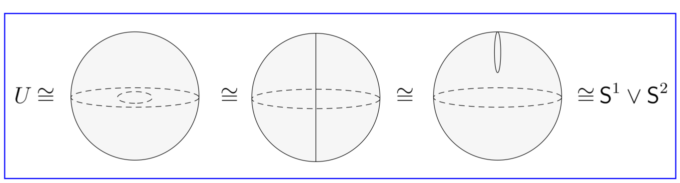
$$\mathsf{deg}(h)\underbrace{\int_T\alpha\wedge\beta}_{\neq0}=\int_S h^*(\alpha\wedge\beta)=\int_S\underbrace{(h^*\alpha)}_{=0}\wedge(h^*\beta)=0.$$

□

Problem 6.

- Let X be the complement of the unlink in S^3. By the homotopy below, we view X as a wedge sum of two copies of U, where U is a solid sphere with a circle removed inside. In U, we first stretch the missing circle until we're left with the surface of the sphere together with a line segment connecting the poles; we then translate the south pole along the surface and onto the north pole to obtain the wedge sum shown.



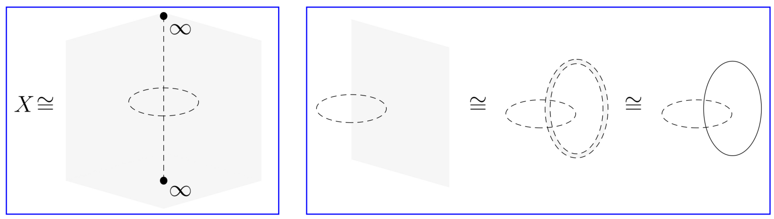


Hence $X\cong U\vee U\cong\mathsf{S}^1\vee\mathsf{S}^1\vee\mathsf{S}^2\vee\mathsf{S}^2$, and so

$$\mathsf{H}_j(X)\cong\mathsf{H}_j(\mathsf{S}^1)^{\oplus2}\oplus\mathsf{H}_j(\mathsf{S}^2)^{\oplus2}\cong\begin{cases}\mathbb{Z} & j=0,\\ \mathbb{Z}^{\oplus2} & j=1,2,\\ 0 & \text{else,}\end{cases}$$

where the $j = 0$ case follows from the fact that X is path connected. $\square$

- Let X be the complement of the Hopf link in S^3. We assume w.l.o.g. that one of the circles passes through ∞, and hence is visualized as a vertical axis in $\mathbb{R}^3$, surrounded by the second circle. Then X is the union of all vertical planes starting at this axis, and each such plane is equivalent to a circle itself, as shown.

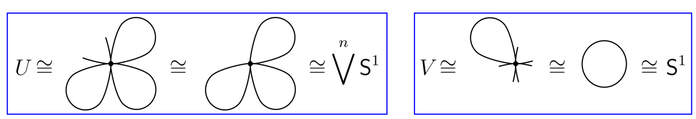


It follows that $X \cong \mathsf{S}^1 \times \mathsf{S}^1 \cong \mathsf{T}^2$, and $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{T}^2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 7.

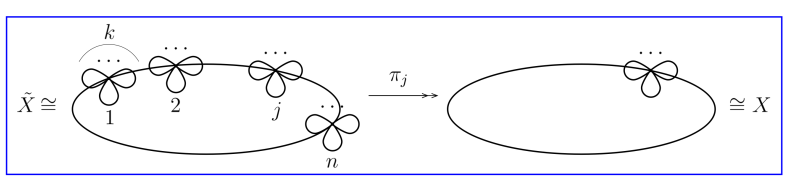
Background. In part (b) we prove the Nielsen-Schreier theorem.

(a) We proceed by induction on n. The case $n = 0$ is immediate since $\pi_1(\mathsf{S}^1) \cong \mathbb{Z} \cong \mathsf{F}_1$, so let $n \geq 1$ be arbitrary and assume $\pi_1\left(\bigvee^n \mathsf{S}^1\right) \cong \mathsf{F}_n$. Defining



gives $U \cup V \cong \bigvee^{n+1} \mathsf{S}^1$ and $U \cap V \cong *$, so $\pi_1\left(\bigvee^{n+1} \mathsf{S}^1\right) \cong \pi_1\left(\bigvee^n \mathsf{S}^1\right) * \pi_1(\mathsf{S}^1) \cong \mathsf{F}_n * \mathsf{F}_1 \cong \mathsf{F}_{n+1}$ by van Kampen. $\square$

(b) Let $X := \bigvee^{n+1} \mathsf{S}^1$. If $H \subset \mathsf{F}_{n+1} \cong \pi_1(X)$ is a subgroup with $[\mathsf{F}_{n+1} : H] = k$, then $H \cong \pi_1(\tilde{X})$ for some k-fold covering space $\tilde{X} \twoheadrightarrow X$. Note that $\tilde{X}$ is a connected graph since it's a covering space of a connected graph, and thus $\tilde{X}$ is homotopy equivalent to a wedge of circles. We observe that the covering space



obtained by attaching k copies of $\bigvee^n \mathsf{S}^1$ to a base circle gives the desired wedge product, and

$$H \cong \pi_1(\tilde{X}) \cong \pi_1\Big(\bigvee^{kn+1} \mathsf{S}^1\Big) \cong \mathsf{F}_{kn+1}.$$

□