Geometry/Topology Qualifying, Spring 2006

Partial credit for partial solutions

1. Let (x, y, z, w) be Cartesian coordinates on $\mathbf{R}^4$. Is the set defined by the equation $x^2 + xy^3 + yz^4 - w^5 = -1$ a smooth manifold of $\mathbf{R}^4$? Prove your assertion.

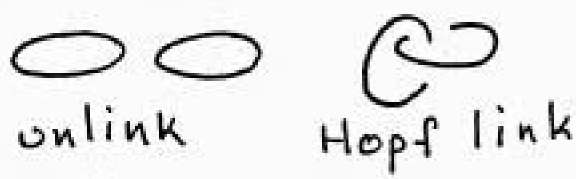
2. **a)** State the definition of the ith de Rham cohomology group $H^i_{dR}(M)$ of a smooth manifold M.
b) Compute the ith de Rham cohomology groups of the real line $\mathbf{R}$ directly from the definition for all $i \geq 0$.

3. Let X be the quotient space obtained from the n-dimensional sphere S^n by identifying three distinct points to a single common point $p \in X$. In other words, let $q, r, s \in S^3$ be pairwise distinct points, let $X = S^n/\sim$ where $x \sim y$ if $x = y$ or if $x, y \in \{q, r, s\}$, and let $p \in X$ denote the equivalence class $\{q, r, s\}$. Calculate $\pi_1(X, p)$.

4. Let $S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbf{R}^4$ and let $\omega = w\, dx \wedge dy \wedge dz$. Compute $\int_{S^3} \omega$.

5. Recall that the *genus* of a closed orientable surface Σ is defined to be $\frac{1}{2}\dim_{\mathbf{R}} H^1_{dR}(\Sigma)$. Let S and T be closed orientable surfaces of respective genera $g(S)$ and $g(T)$. Assume $g(S) < g(T)$. Show that the degree of any smooth map $h : S \to T$ equals zero. [You may use the fact that on a closed orientable surface Σ, the wedge product of one-forms induces a skew-symmetric non-degenerate bilinear pairing $H^1_{dR}(\Sigma) \otimes H^1_{dR}(\Sigma) \to H^2_{dR}(\Sigma) \approx \mathbf{R}$, where $H^i_{dR}(F)$ denotes the ith de Rham cohomology group of Σ.]

6. Define the *unlink* to be the union of two unknotted circles in the three-dimensional sphere S^3, where there are two disjoint three-dimensional balls in S^3 containing the circles. Define the *Hopf link* to be the union of two unknotted disjoint circles in S^3, where each circle meets a disk bounding the other circle in a single point. These links are illustrated in the figure below drawn in $\mathbf{R}^3 = S^3 - \{\text{the point at infinity}\}$. Let U be the complement in S^3 to the unlink and let H be the complement in S^3 to the Hopf link. Calculate the homology groups of U and H.



7. Let X denote a bouquet of $n + 1$ circles, i.e., X is the quotient of the disjoint union of $n + 1$ circles with base points obtained by identifying all the base points to a single point p in the quotient.
a) Prove that $\pi_1(X, p)$ is a free group F_{n+1} on $n + 1$ generators.
b) Let H be a subgroup of F_{n+1} of index k. Show that H is a free group with $kn + 1$ generators.