2

(7.) Let X be the complement of the knot K in the solid torus $S^1 \times D^2$ as in Figure 1. Compute the homology groups $H_i(X; \mathbf{Z})$.

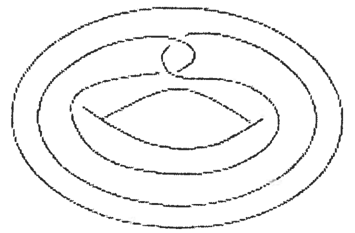


FIGURE 1

1. $S^n \xrightarrow{g} \vee_m S^n \xrightarrow{h} S^n$, g collapses the complement of m B^n's to a point. h maps each S^n to S^n. For all but 1 $y \in S^n$, $f^{-1}(y)$ consists of m points.

2. a) Let $a = \int_{T^2} w_2$. Use $I : H^2_{dR}(T^2) \to \mathbb{R}$ isomorphism. ; Both are questions of dimension of $H^0_{dR}(T^2)$
   b)

3.

4. Regular Value Theorem

5. If $p: \tilde{X} \to X$ is a covering space, $p^*: \pi_n(\tilde{X}) \to \pi_n(X)$ is an isomorphism for all $n \geq 2$.

6. Use definition of K, enclose Σ is smallest sphere, look at point where Σ touches sphere.

7. Inverse Mayer Vietoris

Geo/Top. Spr. 05:

(1.) For each $n>0$, $m \in \mathbb{Z}$, show $\exists\, f: S^n \to S^n$ of degree m

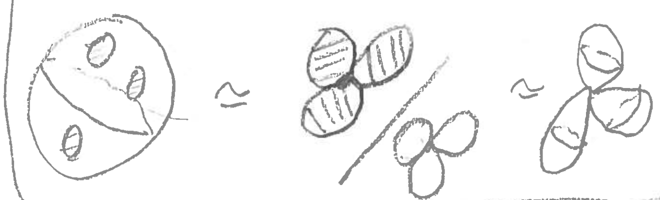
Consider m disjoint copies of B_ε^n contained in S^n. Call them $B_1, \dots, B_m$.

Now see that $S^n/(B_1 \cup \dots \cup B_m)^c \simeq B_1 \vee \dots \vee B_m / \partial B_1 \vee \dots \vee \partial B_m$

$\simeq B_1/\partial B_1 \vee \dots \vee B_m/\partial B_m$

$\simeq S^n \vee \dots \vee S^n$ (m times)

Picture:



So define $g: S^n \to S^n/(B_1 \cup \dots \cup B_m)^c \simeq \vee^m S^n$

as the map collapsing $(B_1 \cup \dots \cup B_m)^c$ to a point

Now define $h: \vee^m S^n \to S^n$ to be the map sending each copy of S^n to S^n via the identity.

Now $h \circ g: S^n \to \vee^m S^n \to S^n$

Now, $h^{-1}(x)$ consists of m points, one in each copy of S^n, and then $g^{-1}(h^{-1}(x))$ is also m distinct points since each point on the distinct S^n's in $\vee^m S^n$ has preimage a point in one of the distinct B_i's. Therefore $\deg(h \circ g) = m$

(for negative m, let h map via antipode)

(2) $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x,y) inherited

(a) For any $\omega_2 \in \Omega^2(T^2)$, $\exists\, \omega_1 \in \Omega^1(T^2)$ & $a \in \mathbb{R}$ s.t.
$\omega_2 = a\,dx \wedge dy + d\omega_1$

Since $\Omega^3(T^2) = 0$, $d\omega_2 = 0$, hence $[\omega_2] \in H^2(T^2)$ well-def.
Since T^2 cpt, orient, the map $I: H^2(T^2) \to \mathbb{R}$, $\omega \longmapsto \int_{T^2} \omega$ is iso.

$H^2(T^2)$ is generated by the volume form $dx \wedge dy$,
hence $[\omega] = \left(\int_{T^2} \omega\right) [dx \wedge dy]$; let $\int_{T^2} \omega = a$; then

$[\omega_2] = [a\,dx \wedge dy]$

Now, $[\omega_2 - a\,dx \wedge dy] = 0$, hence exact, hence
$\exists\, \omega_1 \in \Omega^1(T^2)$ s.t. $d\omega_1 = \omega_2 - a\,dx \wedge dy \Rightarrow \underline{\omega_2 = d\omega_1 + a\,dx \wedge dy}$

(b) Prove for any closed $\omega_1 \in \Omega^1(T^2)$, $\exists$ smooth $f: T^2 \to \mathbb{R}$
and $a, b \in \mathbb{R}$ s.t. $\omega_1 = a\,dx + b\,dy + df$

ω_1 closed $\Rightarrow [\omega_1] \in H^1(T^2)$ well-def.
Now, the classes of the coordinate covector fields dx, dy
form an $\mathbb{R}$-basis for $H^1(T^2)$, hence $\exists\, a, b$ s.t.

$$[\omega_1] = [a\,dx + b\,dy]$$

$\Rightarrow [\omega_1 - a\,dx - b\,dy] = 0 \Rightarrow \omega_1 - a\,dx - b\,dy$ exact
$\Rightarrow \exists f$ s.t. $df = \omega_1 - a\,dx - b\,dy$
$\Rightarrow \underline{\omega_1 = df + a\,dx + b\,dy}$

④ $A \in M_n(\mathbb{R})$, $\det A \neq 0$, $A^T = A$, $c \in \mathbb{R}$ nonzero

Show that $M = \{x \in \mathbb{R}^n : \langle x, Ax \rangle = c\}$ is a submfld

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We'll apply the reg. val. thm.

Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ Then $f^{-1}(c) = M \subseteq \mathbb{R}^n$

$x \longmapsto \langle x, Ax \rangle$

Now must only show T_pf surjective $\forall p \in f^{-1}(c)$.

Consider $v \in T_p\mathbb{R}^n$; then we have representative curve $\alpha(t) = tv + p$. Then $T_pf(v) = (f \circ \alpha)'(0)$

So: $(f \circ \alpha)(t) = f(\alpha(t)) = f(tv+p) = \langle tv+p, A(tv+p) \rangle$

$= \langle tv+p, tAv + Ap \rangle = \langle tv, tAv + Ap \rangle + \langle p, tAv + Ap \rangle$

$= \langle tv, tAv \rangle + \langle tv, Ap \rangle + \langle p, tAv \rangle + \langle p, Ap \rangle$

$= t^2 \langle v, Av \rangle + t \langle v, Ap \rangle + t \langle p, Av \rangle + c$

$\Rightarrow (f \circ \alpha)'(t) = 2t \langle v, Av \rangle + \langle v, Ap \rangle + \langle p, Av \rangle$

$\Rightarrow T_pf(v) = (f \circ \alpha)'(0) = \langle v, Ap \rangle + \langle p, Av \rangle$.

Therefore, $T_pf(p) = \langle p, Ap \rangle + \langle p, Ap \rangle = 2c \neq 0$,

hence T_pf is surjective since codom $= \mathbb{R}$ & $\neq 0$.

(5.) Compute $\pi_2(S^2 \vee S^1)$.

Cf. Ex. 4.26 Hatcher

Recall that if $p: \tilde{X} \to X$ covering space, then $p^*: \pi_n(\tilde{X}) \to \pi_n(X)$ iso. for $n \geq 2$.

So consider $X = S^2 \vee S^1$.
S^2 has universal cover itself & S^1 has universal cover $\mathbb{R}$, hence a universal cover of X is:

e.g., a countable wedge of S^2's.

Therefore we have $\pi_2(S^2 \vee S^1) \cong \pi_2(\vee^\infty S^2)$

(6) $\Sigma \subseteq \mathbb{R}^3$ embedded cpt surface.
Then prove $\exists x \in \Sigma$ st. $K(x) \geq 0$

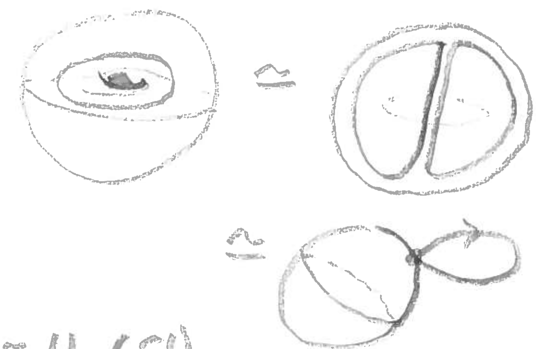
Since Σ is compact, we may cover it with a large enough closed ball B_r^3.
Let r be minimal, hence at least one point of Σ will touch ∂B_r^3, hence that point will be in $\partial B_r^3 \simeq S_r^2$; but every point on S^2 has positive curvature, hence this pt in Σ will have positive curvature!

7. $X = (S^1 \times D^2) \setminus K$ where K pictured:

Find $H_i(X)$

Let $A = \mathbb{R}^3 \setminus K$, $B = S^1 \times D^2$, $A \cap B = X$, $A \cup B = \mathbb{R}^3$

See that $A = \mathbb{R}^3 \setminus K \simeq S^1 \vee S^2$:



and B = solid torus $\simeq S^1$.

So we have:

$H_i(A) \cong H_i(S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$ $\quad H_i(B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$

$H_i(A \cup B) \cong H_i(pt) \cong \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

Now apply Mayer-Vietoris:

$0 \to H_3(X) \to H_3(A) \oplus H_3(B) \to H_3(\underset{pt}{A \cup B}) \to H_2(X) \to H_2(A) \oplus H_2(B)$

$\to H_2(\underset{pt}{A \cup B}) \to H_1(X) \to H_1(A) \oplus H_1(B) \to H_1(\underset{pt}{A \cup B}) \to H_0(X) \to$

$H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$

$\Rightarrow \quad 0 \to H_3(X) \to 0 \to 0 \to H_2(X) \to \mathbb{Z} \to 0 \to H_1(X) \to \mathbb{Z} \oplus \mathbb{Z}$

$\to 0 \to H_0(X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$

- X path connected, hence $\underline{H_0(X) \cong \mathbb{Z}}$
- $H_3(X) = 0$ and $H_2(X) = \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ by exactness.

$$\Rightarrow \boxed{H_i(X) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} \oplus \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}}$$