Geometry/Topology Qualifying Exam

February 2005

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. For each $n > 0$ and every $m \in \mathbf{Z}$, show that there exists a smooth map $f : S^n \to S^n$ of degree m.

2. Let $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ be a 2-dimensional torus with standard Euclidean coordinates (x, y) inherited from $\mathbf{R}^2$.
   (a) Prove that for any 2-form ω_2 on T^2 there is a 1-form ω_1 on T^2 and a real number a such that
   $$\omega_2 = a dx \wedge dy + d\omega_1.$$
   (b) Prove that for any closed 1-form ω_1 on T^2 there is a smooth function f on T^2 and real numbers a, b so that
   $$\omega_1 = a dx + b dy + df.$$

3. Let M be a nonorientable smooth manifold and $i : M \to \mathbf{R}^m$ be an immersion. Define the *normal bundle* $\nu \to M$ to be the set of points (x, v) where $x \in M$ and $v \in \mathbf{R}^m$ is orthogonal to $i_*(T_xM)$ (with respect to the standard Euclidean metric on $\mathbf{R}^m$). Here i_* is the induced map $T_xM \to T_{i(x)}\mathbf{R}^m$ between tangent spaces and we are identifying $T_{i(x)}\mathbf{R}^m$ with $\mathbf{R}^m$.
   (a) Prove that ν can be given the structure of a smooth manifold.
   (b) Is ν an orientable manifold?

4. Let A be a nonsingular symmetric $n \times n$ matrix and c a nonzero real number. (A matrix is *nonsingular* if $\det A \neq 0$ and *symmetric* if $A^T = A$.) Show that
   $$\{x \in \mathbf{R}^n \mid \langle x, Ax \rangle = c\}$$
   is a submanifold of $\mathbf{R}^n$. Here $\langle , \rangle$ is the standard inner product on $\mathbf{R}^n$. What is the dimension of the submanifold?

5. Compute the second homotopy group $\pi_2(S^2 \vee S^1)$ of the wedge sum of S^2 and S^1.

6. Let Σ be an embedded compact surface without boundary in $\mathbf{R}^3$. Then prove that there is a point $x \in \Sigma$ where the Gaussian curvature $K(x)$ is positive. Here the Gaussian curvature is computed with respect to the metric induced from $\mathbf{R}^3$.

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7. Let X be the complement of the knot K in the solid torus $S^1 \times D^2$ as in Figure 1. Compute the homology groups $H_i(X; \mathbf{Z})$.

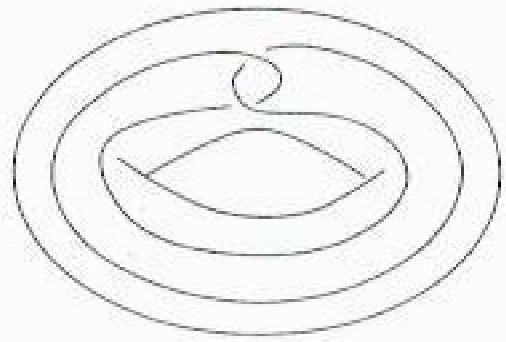

FIGURE 1