Geometry/Topology Qualifying Exam

February 2004

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold of dimension n (without boundary). Let $\omega \in \Omega^n(M)$ be an n-form on M and X a vector field on M. Prove that $\mathcal{L}_X\omega = 0$ at some point $p \in M$. (Here $\mathcal{L}_X\omega$ is the Lie derivative of ω in the direction X.)

2. Let
$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
be a 2-form defined on $\mathbf{R}^3 - \{0\}$. If $i : S^2 = \{x^2+y^2+z^2 = 1\} \to \mathbf{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbf{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$.

3. Consider the set $X \subset \mathbf{R}^4$ defined by the simultaneous equations $x^2 + y^2 - z^2 - w^2 = 1$ and $xz + yw = 1$. Is X a smooth submanifold of $\mathbf{R}^4$?

4. Show that any smooth function $g : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point. Here $\mathbf{RP}^k$ is the *real projective space*, defined as the quotient of the k-dimensional sphere $S^k = \{|x| = 1\} \subset \mathbf{R}^{k+1}$ by the equivalence relation $x \sim -x$.

5. Let $S^1 = \{x^2 + y^2 = 1, z = 0\}$ denote the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$ (where $\mathbf{R}^3$ has standard coordinates (x, y, z)). Calculate the fundamental group of $\mathbf{R}^3 - S^1$.

6. Let X be a connected covering space of the 2-dimensional torus $T^2 = S^1 \times S^1$. List all the possible homeomorphism types of X.

7. For a topological space X, its *suspension* ΣX is the quotient $(X \times [0,1])/\sim$ of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. (More precisely, the equivalence relation $\sim$ is given by:
$$\forall x, x' \in X \ (x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1).)$$
For any $p \geq 2$, prove that $H_p(\Sigma X, \mathbf{Z})$ is isomorphic to $H_{p-1}(X, \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers. What happens when $p = 0, 1$?

1. Stokes, $L_X\omega = d\circ i_X\omega + i_X \circ d\omega = d\circ i_X\omega$
2. Consider the Ellipse $9x^2+4y^2+64z^2 \leq 1$ minus S^2.
3. Regular Value Theorem
4. If $f: S^k \to S^k$ fixes no point, $\deg f = (-1)^{k+1}$. If $f(x) \neq -x \ \forall x \in S^k$, then $f \simeq id \Rightarrow \deg f = 1$. Lifting Criterion.
5. $\mathbb{R}^3 - S^1 \overset{h.e.}{\simeq} S^1 \vee S^2$
6. Cylinder, Torus, $\mathbb{R}^2$
7. Mayer Vietoris

(1.) M cpt, ornt, dim n, $\partial M = \phi$, $\omega \in \Omega^n(M)$, $X \in \mathfrak{X}(M)$.
Prove that $\mathcal{L}_X \omega = 0$ at some point $p \in M$.

Since $\dim M = n$, $\Omega^{n+1}(M) = 0$, hence $d\omega = 0$.
Now recall that

$$\mathcal{L}_X \omega = d \circ i_X(\omega) + i_X \circ \overbrace{d\omega}^{=0} = d(i_X(\omega))$$

we know $i_X(\omega) \in \Omega^{n-1}(M) \Rightarrow d(i_X(\omega)) \in \Omega^n(M)$,

So :

$$\int_M \mathcal{L}_X \omega = \int_M d(i_X(\omega)) \overset{\text{Stokes}}{=} \int_{\partial M} i_X(\omega) = \int_{\phi} i_X(\omega) = 0$$

Hence $\int_M \mathcal{L}_X \omega = 0$, hence $(\mathcal{L}_X \omega)_p = 0$ for some $p \in M$
by Intermediate Value Theorem.

(2) $\omega = \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3\setminus\{0\})$

$i: S^2 \hookrightarrow \mathbb{R}^3$, $j: S^2 \to \mathbb{R}^3$
$(x,y,z) \mapsto (3x, 2y, 8z)$

Compute $\int_{S^2} i^*\omega$ & $\int_{S^2} j^*\omega$:

- $\int_{S^2} i^*\omega = \int_{S^2} \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$

$= \int_{S^2} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$ defined at zero, so may apply Stokes.

$= \int_{B^3} d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$

$= \int_{B^3} dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy$

$= \int_{B^3} 3\,dx\wedge dy\wedge dz = \underline{3\,\mathrm{vol}(B^3)}$.

- $\int j^*\omega = \dfrac{(x\circ j)(d(y\circ j)\wedge d(z\circ j)) + (y\circ j)\,d(z\circ j)\wedge d(x\circ j) + (z\circ j)\,d(x\circ j)\wedge d(y\circ j)}{((x\circ j)^2+(y\circ j)^2+(z\circ j)^2)^{3/2}}$

$= \dfrac{(3x)\,d(2y)\wedge d(8z) + (2y)\,d(8z)\wedge d(3x) + (8z)\,d(3x)\wedge d(2y)}{[(3x)^2+(2y)^2+(8z)^2]^{3/2}}$

$= \dfrac{48(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)}{[9x^2+4y^2+64z^2]^{3/2}}$

Now, $j^*\omega \in \Omega^2(S^2)$ and closed since $\Omega^3(S^2)=0$, hence $[j^*\omega]\in H^2(S^2)$ well-defined. Now consider the ellipsoid $E = \{9x^2+4y^2+64z^2 \le 1\}$; clearly $\partial E \simeq S^2$ (homotopy equiv.), hence $H^2(\partial E) \cong H^2(S^2)$, but recall top dim'l de Rham iso gives iso. $H^2(S^2) \cong \mathbb{R}$ by $\omega \mapsto \int_{S^2}\omega$, hence $\int_{S^2} j^*\omega = \int_{\partial E} j^*\omega$

2 cont'd : So:

$$\int_{S^2} j^*\omega = \int_{\partial E} j^*\omega = \int_{\partial E} \frac{48\,(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)}{(9x^2+4y^2+64z^2)^{3/2}}$$

$$= \int_{\partial E} 48(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) \rightarrow \text{defined at } 0, \text{ so apply Stokes}$$

$$= 48 \int_E d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= 48 \int_E dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy$$

$$= 48 \int_E 3\,dx\wedge dy\wedge dz = \underline{144\,\mathrm{vol}(E)}$$

(3.) $X = \{ x^2+y^2-z^2-w^2=1,\ xz+yw=1 \} \subset \mathbb{R}^4$

Is X smooth mfld?

we'll apply regular value theorem twice.

Let $X_1 = \{x^2+y^2-z^2-w^2=1\}$, hence for $f_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}$
$(x,y,z,w) \longmapsto x^2+y^2-z^2-w^2-1$,

$X_1 = f_1^{-1}(0)$; So choose $p \in f_1^{-1}(0)$ and consider:

$$T_p f_1 : T_p \mathbb{R}^4 \longrightarrow T_0 \mathbb{R}$$

$$T_p f_1 = [2x \ \ 2y \ -2z \ -2w]\Big|_p$$

But $p=(x,y,z,w)$ satisfies $x^2+y^2-z^2-w^2=1$, hence not all coordinates are zero, hence $T_p f_1$ must have one nonzero entry, i.e. $T_p f_1 \neq 0$, hence surjective.

So X_1 is a smooth mfld.

Now let $X = \{(x,y,z,w) \in X_1 : xz+yw=1\} \subseteq X_1 \subseteq \mathbb{R}^4$.
and consider the map $f : X_1 \longrightarrow \mathbb{R}$; again, $f^{-1}(0) = X$.
$(x,y,z,w) \longmapsto xz+yw-1$

So choose $q \in f^{-1}(0)$ and consider $T_q f : T_q X_1 \to T_0 \mathbb{R}$

$$T_q f = [z \ w \ x \ y]\Big|_q$$

But recall that $q \in X_1$, hence again $q=(x,y,z,w)$ satisfies $x^2+y^2-z^2-w^2=1$, hence not all 0, hence $T_q f \neq 0$, hence surjective.

So X is a submfld of submfld X_1, hence a submfld of $\mathbb{R}^4$.

④ Show any smooth fn. $g: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.
$(\mathbb{R}P^{2n} = S^{2n}/_{\alpha(x)\sim x})$

3.

Suppose $g: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has $g(x) \neq x$ for all $x \in \mathbb{R}P^{2n}$.
Now recall the quotient map $\pi: S^{2n} \to \mathbb{R}P^{2n}$
and consider the map $g \circ \pi: S^{2n} \to \mathbb{R}P^{2n}$.

Then $(g\circ\pi)^*(\pi_1(S^{2n})) = (g\circ\pi)^*(1) = 1 \leq \pi^*(\pi_1(S^{2n})) = 1$,
hence lifting criterion is satisfied and we get $\tilde{g}$!

$$\begin{tikzcd} & S^{2n} \arrow[d, "\pi"] \\ S^{2n} \arrow[ur, dashed, "\tilde{g}"] \arrow[r, "g\circ\pi"'] & \mathbb{R}P^{2n} \end{tikzcd}$$

$g\circ\pi : S^{2n} \longrightarrow \mathbb{R}P^{2n}$

i.e. $\pi\circ\tilde{g} = g\circ\pi$

$\Rightarrow \pi\circ\tilde{g}(p) = g\circ\pi(p)$

$\Rightarrow [\tilde{g}(p)] = g([p]) \neq [p]$ g has no fixed pts by hypothesis

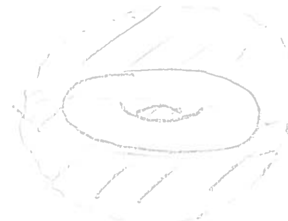
hence $\tilde{g}(p) \neq p$ or $-p$ for all $p \in S^{2n}$,

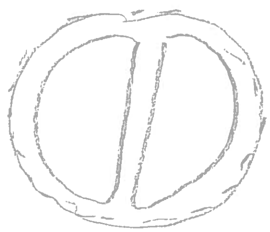
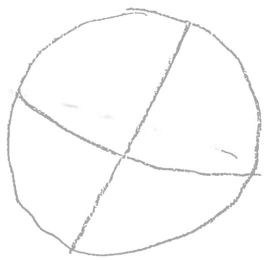
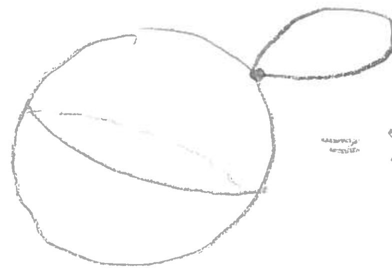
but then $\tilde{g}(p) \neq p \ \forall p \Rightarrow \deg\tilde{g} = \deg\alpha = (-1)^{2n+1} = -1$
$\tilde{g}(p) \neq -p \ \forall p \Rightarrow \deg\tilde{g} = \deg id = 1$,

a contradiction, hence g must have had a fixed point

5.) $\pi_1(\mathbb{R}^3\setminus S^1)$:

See the following:

$\mathbb{R}^3\setminus S^1 =$  $\simeq$ 

$\simeq$  $\simeq$  $\simeq$  $\simeq S^1 \vee S^2$

Hence $\pi_1(\mathbb{R}^3\setminus S^1) \cong \pi_1(S^1\vee S^2) \underset{\text{Seifert-van Kampen}}{\cong} \pi_1(S^1) * \pi_1(S^2) = \mathbb{Z} * 0$

$= \mathbb{Z}$

(6.) List the homeomorphism classes of connected covers of $T^2 = S^1 \times S^1$:

Recall the classification of connected covers gives 1-1 Galois correspondence between subgroups of $\pi_1(T^2)$ and the connected covering spaces.

See that $\pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

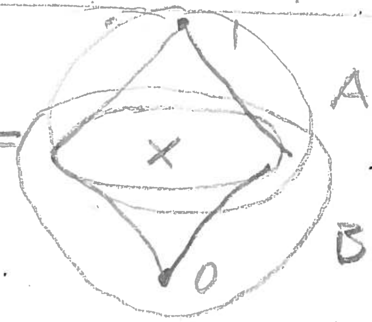
Now we have:

Subgroup	Cover
$0 \oplus 0$	$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ (universal cover)
$0 \oplus \mathbb{Z}$	$\mathbb{R} \times S^1 \simeq$ cylinder
$\mathbb{Z} \oplus 0$	$S^1 \times \mathbb{R} \simeq$ cylinder
$\mathbb{Z} \oplus \mathbb{Z}$	T^2 (trivial cover)

Note: if $\mathbb{Z} = \langle \alpha \rangle$, then subgroup $\langle \alpha^2 \rangle \cong \mathbb{Z}$, hence subgroups like $0 \oplus \langle \alpha^2 \rangle$ still correspond to covering space $\mathbb{R} \times S^1$, but w/ different # of sheets (2 here).

(7.) X top space, $\Sigma X = (X \times [0,1]) \Big/ \begin{smallmatrix}(x,0)\sim(x',0) \\ (x,1)\sim(x',1)\end{smallmatrix} \ \forall x,x'$

For $p \geq 2$, prove $H_p(\Sigma X) \cong H_{p-1}(X)$. What about $p = 0, 1$?

See that $\Sigma X =$ 

Let A, B be as pictured; then $A \simeq pt$, $B \simeq pt$, $A \cap B \simeq X$, and $A \cup B = \Sigma X$. Now apply Mayer-Vietoris ($p \geq 2$)

$$\dashrightarrow H_p(A) \oplus H_p(B) \to H_p(\Sigma X) \to H_{p-1}(X) \longrightarrow H_{p-1}(A) \oplus H_{p-1}(B) \to$$

$$\Rightarrow \quad 0 \to H_p(\Sigma X) \to H_{p-1}(X) \to 0 \Rightarrow \underline{H_p(\Sigma X) \cong H_{p-1}(X)}$$

for $p \geq 2$ by exactness

For $p = 0$, we have $\underline{H_0(\Sigma X) \cong \mathbb{Z}}$ since ΣX path-connected by construction.

For $p = 1$, consider the end of the Mayer-Vietoris sequence:

$$0 \to H_1(\Sigma X) \xrightarrow{\gamma} H_0(X) \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(\Sigma X)}} \to 0$$

By exactness, α is surjective, hence $\ker \alpha = \mathbb{Z} \Rightarrow \operatorname{im} \beta = \mathbb{Z}$

& by exactness, γ is injective; have $H_1(\Sigma X) \cong \operatorname{im} \gamma = \ker \beta$,

i.e. $H_0(X) \cong H_1(\Sigma X) \oplus \mathbb{Z}$, i.e. $\underline{H_1(\Sigma X) \cong \tilde{H}_0(X)}$