Geometry/Topology Qualifying Exam

February 2004

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold of dimension n (without boundary). Let $\omega \in \Omega^n(M)$ be an n-form on M and X a vector field on M. Prove that $\mathcal{L}_X\omega = 0$ at some point $p \in M$. (Here $\mathcal{L}_X\omega$ is the Lie derivative of ω in the direction X.)

2. Let
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
be a 2-form defined on $\mathbf{R}^3 - \{0\}$. If $i : S^2 = \{x^2+y^2+z^2 = 1\} \to \mathbf{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbf{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$.

3. Consider the set $X \subset \mathbf{R}^4$ defined by the simultaneous equations $x^2+y^2-z^2-w^2 = 1$ and $xz + yw = 1$. Is X a smooth submanifold of $\mathbf{R}^4$?
4. Show that any smooth function $g : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point. Here $\mathbf{RP}^k$ is the *real projective space*, defined as the quotient of the k-dimensional sphere $S^k = \{|x| = 1\} \subset \mathbf{R}^{k+1}$ by the equivalence relation $x \sim -x$.
5. Let $S^1 = \{x^2+y^2 = 1, z = 0\}$ denote the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$ (where $\mathbf{R}^3$ has standard coordinates (x, y, z)). Calculate the fundamental group of $\mathbf{R}^3 - S^1$.
6. Let X be a connected covering space of the 2-dimensional torus $T^2 = S^1 \times S^1$. List all the possible homeomorphism types of X.
7. For a topological space X, its *suspension* ΣX is the quotient $(X \times [0,1])/\sim$ of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. (More precisely, the equivalence relation $\sim$ is given by:
$$\forall x, x' \in X \ (x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1).)$$
For any $p \geq 2$, prove that $H_p(\Sigma X, \mathbf{Z})$ is isomorphic to $H_{p-1}(X, \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers. What happens when $p = 0, 1$?