535a/540

Geometry/Topology Qualifying Exam

February 2003

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold M of dimension $2n$ (without boundary), and let ω be a *symplectic form* on M, namely a closed differential form of degree 2 whose n-th exterior power $\omega \wedge \omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove that the second de Rham cohomology $H^2_{dR}(M; \mathbf{R}) \neq 0$ by showing that ω is not exact.

2. Show that the set $Sl(n, \mathbf{R})$ of $n \times n$ matrices A with entries in the real numbers and which satisfy $\det(A) = 1$ is a manifold. What is its dimension?

3. On $\mathbf{R}^4$ with coordinates x_1, y_1, x_2, y_2, consider the 2-form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a smooth function f on $\mathbf{R}^4$, let X be the vector field
$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}.$$
Then compute $\mathcal{L}_X\omega$, the Lie derivative of ω in the direction X.

check

4. Let M be a compact oriented n-dimensional manifold (without boundary), where $n > 1$. Show that there exists a differentiable map $f : M \to S^n$ of degree 1.

5. Recall that two coverings $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *equivalent* if there exists a homeomorphism $\varphi : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. When X is the 2-dimensional torus $S^1 \times S^1$, determine the number of equivalence classes of all coverings $p : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 3 points (for an arbitrary x_0).

6. Compute the homology groups $H_n(X; \mathbf{Z})$ of the complement $X = \mathbf{R}^5 - A$ of a subset $A \subset \mathbf{R}^5$ consisting of 4 points.

7. Let B^n be the closed unit ball in $\mathbf{R}^n$, and let S^{n-1} be its boundary, namely the $(n-1)$-dimensional sphere. If $f : B^n \to \mathbf{R}^n$ is a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$, show that the image $f(B^n)$ contains the ball B^n.

1. Stokes, $\int$ volume form $\neq 0$
2. Regular Value Theorem
3. $\mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X\alpha\wedge\beta + \alpha\wedge\mathcal{L}_X\beta$, $\mathcal{L}_X = d\circ i_X + i_X\circ d$, $i_X(\omega) = \omega(X)$ for 1-forms
4. Use a cutoff function, a nbhd at $x \in M$, and a buffer nbhd.
5. n sheeted covering spaces $\overset{1:1}{\longleftrightarrow}$ $\rho : \pi_1(X) \to S_n/\sim$ where $\rho_1 \sim \rho_2$ iff $\rho = h\rho_2 h^{-1}$
6. $\mathbb{R}^5 - A \overset{h.e.}{\simeq} \bigvee_{i=1}^{4} S^4$
7. There is no retract $r : B^n \to S^{n-1}$.

Geo/Top - Sp'03 : 1.

(1.) M cpt, ornt, dim $2n$, $\partial M = \emptyset$, $\omega \in \Omega^2(M)$ with $d\omega = 0$ and $\omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form.
Prove that $H^2_{dR}(M) \neq 0$ by showing ω not exact

Suppose ω is exact, i.e. $\exists\, \beta \in \Omega^1(M)$ s.t. $d\beta = \omega$.

Then $\int_M \omega \wedge \cdots \wedge \omega = \int_M d\beta \wedge \cdots \wedge d\beta = \int_M d(\beta \wedge d\beta \wedge \cdots \wedge d\beta)$

$= \int_{\partial M} \beta \wedge d\beta \wedge \cdots \wedge d\beta = \int_{\emptyset} \beta \wedge d\beta \wedge \cdots \wedge d\beta = 0$, a contradiction

since $\omega \wedge \cdots \wedge \omega$ is a volume form

[note that $d(\beta \wedge d\beta \wedge \cdots \wedge d\beta) = d\beta \wedge \cdots \wedge d\beta - \beta \wedge \underbrace{d(d\beta \wedge \cdots \wedge d\beta)}_{0}$].

Therefore, since $d\omega = 0$, $[\omega] \in H^2(M)$ well-defined,
but ω not exact, so $[\omega] \neq 0$, hence $H^2(M) \neq 0$.

(2) Show $SL_n = \{A \in M_n(\mathbb{R}) : \det A = 1\}$ is a manifold.
Dimension?

Consider the function $f : M_n(\mathbb{R}) \to \mathbb{R}$
$$A \mapsto \det A$$

Then $f^{-1}(1) = SL_n$, hence must only show 1 is a regular value. Choose $P \in f^{-1}(1)$ and consider

$$T_p f : T_p M_n(\mathbb{R}) \longrightarrow T_1 \mathbb{R}$$

$$T_p f = \left[\frac{\partial f}{\partial x_{11}} \; \frac{\partial f}{\partial x_{12}} \cdots \frac{\partial f}{\partial x_{nn}} \right] \Big|_{p} =$$

Now, $f(X) = f(x_{ij}) = \det(x_{ij}) = \sum_{j=1}^{n} (-1)^{j+1} x_{1j} \det(X_{1j})$

Then $\dfrac{\partial f}{\partial x_{1j}} = (-1)^{j+1} \det(X_{1j})$ so:

$$T_p f = \left[\det(P_{11}) \; -\det(P_{12}) \; \det(P_{13}) \cdots (-1)^{n+1} \det(X_{1n}) \; \cdots \right]$$

but since $\det(P) = 1 \neq 0$, not all of $\det(P_{1j})$ can be zero, hence $T_p f \neq 0$, hence $T_p f$ surjective and SL_n manifold of dimension $n^2 - 1$.

(3) $\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}$, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \in \Omega^2(\mathbb{R}^4)$

Given $f: \mathbb{R}^4 \to \mathbb{R}$, let X be given by

2.

$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}$$

Compute $\mathcal{L}_X \omega$:

$$\left\{\begin{aligned} \text{Recall } \mathcal{L}_X(\alpha\wedge\beta) &= \mathcal{L}_X\alpha\wedge\beta + \alpha\wedge\mathcal{L}_X\beta \\ i_X(\alpha\wedge\beta) &= i_X\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge i_X\beta \\ i_X(\omega) &= \omega(X) \text{ for } \omega\in\Omega^1 \end{aligned}\right\}$$

Now: $\mathcal{L}_X\omega = \mathcal{L}_X(dx_1\wedge dy_1 + dx_2\wedge dy_2)$

$$= \mathcal{L}_X(dx_1\wedge dy_1) + \mathcal{L}_X(dx_2\wedge dy_2)$$

$$= (\mathcal{L}_X dx_1\wedge dy_1) + (dx_1\wedge\mathcal{L}_X dy_1) + (\mathcal{L}_X(dx_2)\wedge dy_2) + (dx_2\wedge\mathcal{L}_X dy_2)$$

And:

$$\mathcal{L}_X dx_1 = d\circ i_X(dx_1) + i_X\circ \overset{0}{\cancel{d(dx_1)}} = d(dx_1(X)) = d\left(\frac{\partial f}{\partial y_1}\right)$$

$$= \frac{\partial^2 f}{\partial y_1\partial x_1}dx_1 + \frac{\partial^2 f}{\partial y_1^2}dy_1 + \frac{\partial^2 f}{\partial y_1\partial x_2}dx_2 + \frac{\partial^2 f}{\partial y_1\partial y_2}dy_2$$

$$\mathcal{L}_X dy_1 = d\circ i_X(dy_1) = d(dy_1(X)) = d\left(-\frac{\partial f}{\partial x_1}\right)$$

$$= -\frac{\partial^2 f}{\partial x_1^2}dx_1 - \frac{\partial^2 f}{\partial x_1\partial y_1}dy_1 - \frac{\partial^2 f}{\partial x_1\partial x_2}dx_2 - \frac{\partial^2 f}{\partial x_1\partial y_2}dy_2$$

$$\mathcal{L}_X dx_2 = d(dx_2(X)) = d\left(\frac{\partial f}{\partial y_2}\right)$$

$$= \frac{\partial^2 f}{\partial y_2\partial x_1}dx_1 + \frac{\partial^2 f}{\partial y_2\partial y_1}dy_1 + \frac{\partial^2 f}{\partial y_2\partial x_2}dx_2 + \frac{\partial^2 f}{\partial y_2\partial y_2}dy_2$$

$$\mathcal{L}_X dy_2 = d(dy_2(X)) = d\left(-\frac{\partial f}{\partial x_2}\right)$$

$$= -\left(\frac{\partial^2 f}{\partial x_2\partial x_1}dx_1 + \frac{\partial^2 f}{\partial x_2\partial y_1}dy_1 + \frac{\partial^2 f}{\partial x_2^2}dx_2 + \frac{\partial^2 f}{\partial x_2\partial y_2}dy_2\right)$$

So now:

$$\mathcal{L}_X\omega = d\left(\frac{\partial f}{\partial y_1}\right)\wedge dy_1 + dx_1 \wedge d\left(\frac{\partial f}{\partial x_1}\right) + d\left(\frac{\partial f}{\partial y_2}\right)\wedge dy_2 + dx_2\wedge d\left(\frac{\partial f}{\partial x_2}\right)$$

$$= d\left(\frac{\partial f}{\partial y_1}\right)\wedge dy_1 + d\left(\frac{\partial f}{\partial x_1}\right)\wedge dx_1 + d\left(\frac{\partial f}{\partial y_2}\right)\wedge dy_2 + d\left(\frac{\partial f}{\partial x_2}\right)\wedge dx_2$$

$$= \left(f_{y_1x_1}dx_1 + \cancel{f_{y_1y_1}dy_1} + f_{y_1x_2}dx_2 + f_{y_1y_2}dy_2\right)\wedge dy_1$$

$$+ \left(\cancel{f_{x_1x_1}dx_1} + f_{x_1y_1}dy_1 + f_{x_1x_2}dx_2 + f_{x_1y_2}dy_2\right)\wedge dx_1$$

$$+ \left(f_{y_2x_1}dx_1 + f_{y_2y_1}dy_1 + f_{y_2x_2}dx_2 + \cancel{f_{y_2y_2}dy_2}\right)\wedge dy_2$$

$$+ \left(f_{x_2x_1}dx_1 + f_{x_2y_1}dy_1 + \cancel{f_{x_2x_2}dx_2} + f_{x_2y_2}dy_2\right)\wedge dx_2$$

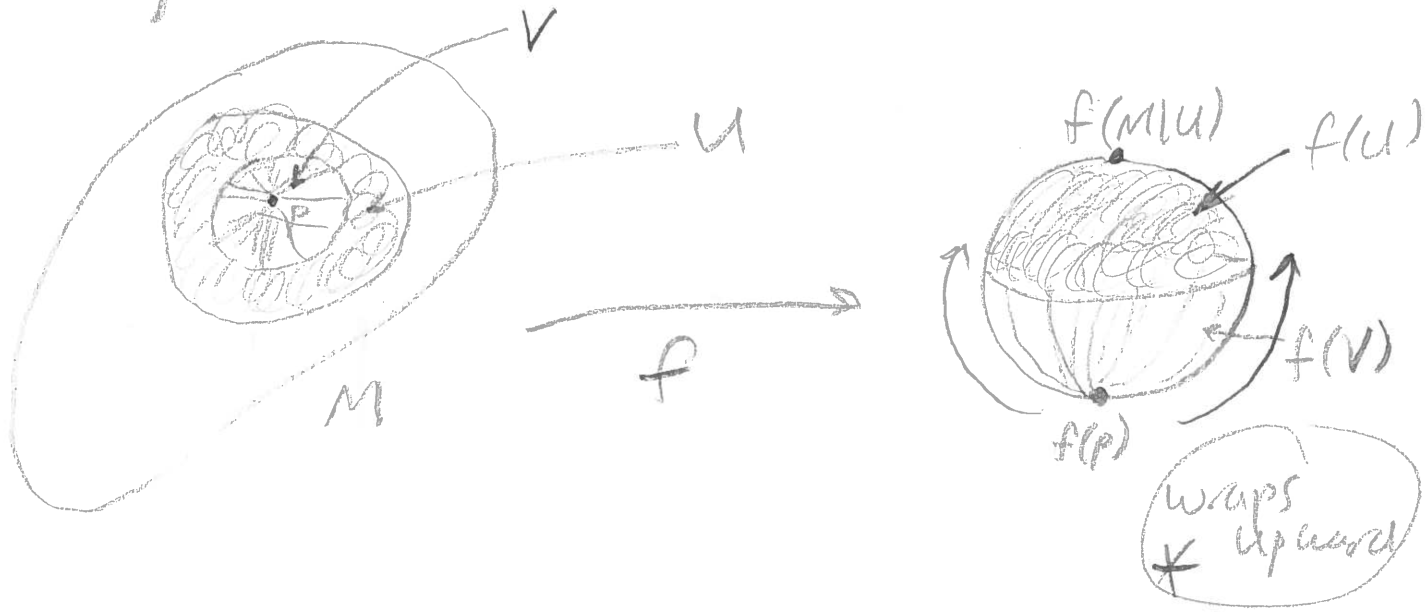
$$= (f_{y_1x_1} - f_{x_1y_1})\,dx_1\wedge dy_1 + (f_{y_2x_1} - f_{x_1y_2})\,dx_1\wedge dy_2$$

$$+ (f_{x_2x_1} - f_{x_1x_2})\,dx_1\wedge dx_2 + (f_{y_2y_1} - f_{y_1y_2})\,dy_1\wedge dy_2$$

$$+ (f_{x_2y_1} - f_{y_1x_2})\,dy_1\wedge dx_2 + (f_{y_2x_2} - f_{x_2y_2})\,dx_2\wedge dy_2$$

4. M cpt, ornt, $n$dim mfld, $\partial M = \emptyset$, $n \geq 1$.
Show $\exists\, f: M \to S^n$ of degree 1

Choose $p \in M$, and cpt neighborhoods $p \in V \subseteq U$.
Then define $f(p) = s \in S^n$, the south pole,
$f(V) = S \subseteq S^n$ the southern hemisphere, $f(U) = N \subseteq S^n$
the northern hemisphere, and $f(M \setminus U) = n \in S^n$, the
north pole:



(5.) $X = T^2 = S^1 \times S^1$; determine the equivalence classes of all 3-sheeted covering spaces.

Recall that 3-sheeted covering spaces are in correspondence with homomorphisms $f: \pi_1(T^2) \to S_3$,

$\pi_1(T^2) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z}\langle\alpha\rangle \oplus \mathbb{Z}\langle\beta\rangle$; homomorphisms are determined by their action on generators, so consider $f(\alpha), f(\beta)$.

Since $\pi_1(T^2)$ is abelian, $f(\alpha)f(\beta) = f(\alpha\beta) = f(\beta\alpha) = f(\beta)f(\alpha)$, hence they must map to commuting permutations. See that $S_3 = \{id, (12), (13), (23), (123), (123)^2 = (132)\}$

No 2-cycle commutes with a 3-cycle, hence we have the following possibilities:

	$f(\alpha)$	$f(\beta)$
7	id	id
	(12)	id
	(13)	id
	(23)	id
	id	(12)
	id	(13)
	id	(23)
4	(123)	id
	$(123)^2$	id
	id	(123)
	id	$(123)^2$

	$f(\alpha)$	$f(\beta)$
7	(12)	(12)
	(23)	(23)
	(13)	(13)
	(123)	(123)
	(132)	(132)
	(123)	(132)
	(132)	(123)

Total = 18

6.) $H_i(\mathbb{R}^5 \setminus \{p_1, p_2, p_3, p_4\}) = ?$

Recall that $\mathbb{R}^5 \setminus \{p_1, p_2, p_3, p_4\} \simeq \bigvee_{i=1}^{4} S^4$,

and the wedge of the spheres has S^4 forming a good pair with the basepoint for all 4 copies,

hence:

$$\tilde{H}_i\left(\bigvee_{i=1}^{4} S^4\right) = \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4)$$

$$= \begin{cases} 0 & 0 \le i \le 3,\ i > 4 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & i = 4 \end{cases}$$

Recall $\tilde{H}_i(\bigvee^4 S^4) \cong H_i(\bigvee^4 S^4) \quad \forall i > 0$

and since $\bigvee^4 S^4$ is path-connected, $H_0(\bigvee^4 S^4) \cong \mathbb{Z}$.

so:

$$\boxed{H_i(\bigvee^4 S^4) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1, 2, 3,\ i > 4 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & i = 4 \end{cases}}$$

(7.) $B^n \subseteq \mathbb{R}^n$ closed unit ball, $\partial B^n = S^{n-1}$

If $f: B^n \to \mathbb{R}^n$ s.t. $f(x) = x$ for all $x \in S^{n-1}$, show that $f(B^n)$ contains B^n.

Suppose $B^n \not\subseteq f(B^n)$, i.e. suppose $p \in B^n$ and $p \notin f(B^n)$, WLOG, let $p = 0$ (can translate for any other point). Then $f(B^n) \subseteq \mathbb{R}^n \setminus \{0\}$. Now consider the following maps:

$$\begin{array}{ccccccc} S^{n-1} & \xhookrightarrow{i} & B^n & \xrightarrow{f} & \mathbb{R}^n \setminus \{0\} & \xrightarrow{r} & S^{n-1} \\ x & \longmapsto & \underset{\in S^{n-1}}{x} & \longmapsto & \underset{\in S^{n-1}}{f(x) = x} & \longmapsto & \frac{x}{\|x\|} = x \end{array}$$

$\|x\| = 1$ since $x \in S^{n-1}$

Therefore $r \circ f \circ i \equiv id_{S^{n-1}}$, hence $(r \circ f \circ i)^*: H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity map.

But note that $i^*: H_{n-1}(S^{n-1}) \to H_{n-1}(B^n)$ is the zero map since a $(n-1)$-cycle in S^{n-1} is clearly an $(n-1)$-boundary in B^n, so $i^* \equiv 0$, which is a contradiction since we have

$$id^* = (r \circ f \circ i)^* = r^* \circ f^* \circ i^* = r^* \circ f^*(0) \equiv 0$$

and $id^* \neq 0$ clearly.

Hence each point of B^n must be in $f(B^n)$, i.e. $B^n \subseteq f(B^n)$.