535a/540

Geometry/Topology Qualifying Exam

February 2003

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold M of dimension $2n$ (without boundary), and let ω be a *symplectic form* on M, namely a differential form of degree 2 whose n–th exterior power $\omega \wedge \omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove that the second de Rham cohomology $H^2_{dR}(M;\mathbf{R}) \neq 0$ by showing that ω is not exact.

2. Show that the set $Sl(n,\mathbf{R})$ of $n \times n$ matrices A with entries in the real numbers and which satisfy $\det(A) = 1$ is a manifold. What is its dimension?

3. On $\mathbf{R}^4$ with coordinates x_1, y_1, x_2, y_2, consider the 2-form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a smooth function f on $\mathbf{R}^4$, let X be the vector field
$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}.$$
Then compute $\mathcal{L}_X\omega$, the Lie derivative of ω in the direction X.

4. Let M be a compact oriented n-dimensional manifold (without boundary), where $n > 1$. Show that there exists a differentiable map $f : M \to S^n$ of degree 1.

5. Recall that two coverings $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *equivalent* if there exists a homeomorphism $\varphi : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. When X is the 2-dimensional torus $S^1 \times S^1$, determine the number of equivalence classes of all coverings $p : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 3 points (for an arbitrary x_0).

6. Compute the homology groups $H_n(X;\mathbf{Z})$ of the complement $X = \mathbf{R}^5 - A$ of a subset $A \subset \mathbf{R}^5$ consisting of 4 points.

7. Let B^n be the closed unit ball in $\mathbf{R}^n$, and let S^{n-1} be its boundary, namely the $(n-1)$–dimensional sphere. If $f : B^n \to \mathbf{R}^n$ is a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$, show that the image $f(B^n)$ contains the ball B^n.