Graduate Exam in Topology/Geometry

February 2002

1. Let $P^n(\mathbb{R})$ be the projective n–space, namely the quotient space of the sphere S^n by the equivalence relation $\sim$ defined by $x \sim y \Leftrightarrow x = \pm y$.
   (a) Show that $P^n(\mathbb{R})$ is a manifold.
   (b) Show that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

2. In the set $M(n)$ of all $n \times n$ matrices, identified to $\mathbb{R}^{n^2}$, consider the subset $O(n)$ consisting of the orthogonal matrices, namely those matrices A for which AA^t is the identity (where A^t denotes the transpose). Show that $O(n)$ is a submanifold of $M(n) = \mathbb{R}^{n^2}$, and that the tangent space $T_{\mathrm{Id}}O(n)$ at the identity Id is equal to the space of all antisymmetric matrices (namely those matrices for which $A^t = -A$).

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (\alpha x + \beta y, \gamma x + \delta y, \varepsilon z)$, where α, β, γ, δ, ε are constants with $\alpha\delta - \beta\gamma = 1$. Find the matrix of $f^* : \Lambda^2\mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ associated to the basis $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$.

4. Let $P^2(\mathbb{R})$ be the real projective plane.
   (a) If $x \in P^2(\mathbb{R})$, compute the fundamental group $\pi_1(P^2(\mathbb{R}) - \{x\})$.
   (b) Show that any map $f : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which is not surjective is homotopic to a constant map. (Hint: use a covering space).

5. Let B^2 be the closed 2–dimensional ball, with boundary the circle S^1. Let $X = S^1 \times B^2$ and let $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_n(X, \partial X)$ with coefficients in Z. (You are allowed to use whatever you may know about the homology of the torus ∂X).

6. Let X be the figure eight  , union of two circles C_1 and C_2 meeting in one point. Let $p : \tilde{X} \to X$ be a covering space such that $\tilde{X}$ is connected and such that the preimage $p^{-1}(x)$ of each $x \in X$ consists of 2 points. Compute the fundamental group of $\tilde{X}$.

7. What are the compact connected surfaces S for which there exists an immersion $S \to S$ which is not a diffeomorphism? (Hint: Euler characteristic).

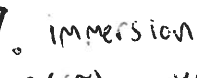
1. a) $\mathbb{Z}_2$ acts on S^n freely, discontinuously. b) HW problem, top dimensional form α can be written as $\alpha = f\omega$ ω volume form.

2. Regular Value Theorem w/ $T_xM = \ker T_xf$.

3. $f^*\omega = \sum_{i_1<\dots<i_k} g_{i_1\dots i_k} \circ f \; df_{i_1} \wedge \dots \wedge df_{i_k}$ where $f = (f_1, \dots, f_n)$ $\omega = \sum_{i_1<\dots<i_k} g_{i_1\dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$

4. a) $P^2(\mathbb{R}) - \{x\}$ def. ret. to S^1 b) f^* is trivial, lifting criterion

5. Long Exact Sequence for Relative Homology

6. 

has fund. group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

7. immersion $\Rightarrow$ covering map (surjective by connectivity) if $p: \tilde{X} \to X$ is n-sheeted,
$\chi(\tilde{X}) = n\chi(X)$, so $\chi(S) = 0$ $\Rightarrow$ S = Torus or Klein Bottle
$\chi(M_g) = 2 - 2g$ $\chi(N_g) = 2 - g$.

Geo/Top - Spring 02

(1.) $\mathbb{R}P^n = S^n / x \sim \alpha(x)$

(a) Show $\mathbb{R}P^n$ is smooth mfld

(b) Show $\mathbb{R}P^n$ orient. $\Leftrightarrow$ n odd

(a) See that since $\langle\alpha\rangle \cong \mathbb{Z}_2$, we have that $\mathbb{R}P^n = S^n/\mathbb{Z}_2$,

i.e. we must only show the action $\mathbb{Z}_2 \curvearrowright S^n$ is free and discontinuous

- Free: α is the only non-identity element in $\mathbb{Z}_2$, and clearly $\alpha(p) = -p \neq p$ for all $p \in S^n$, hence $\mathbb{Z}_2 \curvearrowright S^n$ is free.

- Discontinuous: clearly for any cpt $K \subseteq S^n$ the set $\{\gamma \in \mathbb{Z}_2 : \gamma K \cap K \neq \phi\}$ is finite since $|\mathbb{Z}_2| = 2 < \infty$, so action is trivially discontinuous

$\mathbb{Z}_2 \curvearrowright S^n$ free & discont $\Rightarrow S^n/\mathbb{Z}_2 = \mathbb{R}P^n$ is a manifold

(b) $\mathbb{R}P^n$ ornt $\Leftrightarrow$ n odd.

($\Rightarrow$) $\mathbb{R}P^n$ orientable. This induces an orientation on S^n via quotient $\pi: S^n \to \mathbb{R}P^n$ by letting $(v_1, \dots, v_n) \in T_pS^n$ be positively-ornt. iff $(T_p\pi(v_1), \dots, T_p\pi(v_n))$ positively oriented.

Now recall that $\alpha(v_i) = v_i$ on $\mathbb{R}P^n$ since α identity map on $\mathbb{R}P^n$.

hence $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (T_p\pi(\alpha(v_1)), \dots, T_p\pi(\alpha(v_n)))$,

i.e. $(v_1, \dots, v_n)$ & $(\alpha(v_1), \dots, \alpha(v_n))$ positively oriented on S^n

$\Rightarrow \alpha$ orientation preserving $\Rightarrow$ n odd

($\Leftarrow$) n odd. Then $\alpha: S^n \to S^n$ is orientation preserving.

Now define $(w_1, \dots, w_n) \subseteq T_{\pi(p)} \mathbb{R}P^n$ to be $+$-ont. if $\exists$ $+$-ont basis $(v_1, \dots, v_n) \subseteq T_p S^n$ s.t. $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (w_1, \dots, w_n)$ — Define

well-def. Now suppose $(v_1, \dots, v_n)$ & $(v_1', \dots, v_n')$ are ont. basis for $T_p S^n$ & $T_{-p} S^n = T_{\alpha(p)} S^n$ (resp.) in the same ont class $(+)$.

Now since $\pi \circ \alpha = \pi$,

$$T_p \pi(v_i) = T_p(\pi \circ \alpha)(v_i) = T_{\alpha(p)} \pi \circ T_p \alpha(v_i) = T_{-p} \pi(T_p \alpha(v_i))$$

But α orientation preserving, hence $T_p\alpha(v_i)$ & v_i' are in the same orientation class in $T_{-p} S^n$, hence $T_p \pi(v_i)$ & $T_{-p} \pi(v_i')$ are in the same orientation class in $T_{\pi(p)} \mathbb{R}P^n$.

Hence only $+$-class ont bases may be mapped to what we defined as the $+$-class ont basis for $T_{\pi(p)} \mathbb{R}P^n$

i.e. our ont. is well-defined

(2.) Show $O_n = \{A \in M_n : AA^T = I\} \subseteq M_n$ submfld
and $T_I(O_n) = AntiSym_n$

Let $f: M_n \to Sym_n$ where $Sym_n = \{A^T = A\}$
$X \mapsto XX^T$

(since $\operatorname{im} f = \{XX^T\}$ and $(XX^T)^T = XX^T$, we have $\operatorname{im} f \subseteq Sym_n$)

So $f^{-1}(I) = O_n$; now consider $P \in f^{-1}(I)$ and :

$$T_P f : T_P M_n \to T_I Sym_n$$

For $A \in T_P M_n$, let $\alpha(t) = At + P$ be representative curve ;
then $T_P f(A) = (f \circ \alpha)'(0)$:

$(f \circ \alpha)(t) = f(At+P) = (At+P)(At+P)^T = (At+P)(A^T t + P^T)$
$= AA^T t^2 + AP^T t + PA^T t + PP^T$
$\Rightarrow (f \circ \alpha)'(t) = 2AA^T t + AP^T + PA^T$
$\Rightarrow T_P f(A) = (f \circ \alpha)'(0) = AP^T + PA^T$.

Now, recall that $P \in f^{-1}(I)$, hence $PP^T = I$; now
consider any symmetric matrix $C \in T_I Sym_n \cong Sym_n$
Then: $\frac{1}{2} CP \in T_P M_n \cong M_n$ and:

$T_P f(\frac{1}{2} CP) = (\frac{1}{2} CP) P^T + P(\frac{1}{2} CP)^T$
$= \frac{1}{2} C + \frac{1}{2} PP^T C^T = \frac{1}{2} C + \frac{1}{2} C^T = \frac{1}{2} C + \frac{1}{2} C = C$,
hence $T_P f$ is surjective for all $P \in f^{-1}(I)$, so O_n submfld

Now recall that $T_I O_n = \ker T_I f = \{A : T_I f(A) = 0\}$
$= \{A : A + A^T = 0\} = \{A = -A^T\} = \underline{AntiSym_n}$

(3) $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x,y,z) = (\alpha x + \beta y,\ \gamma x + \delta y,\ \varepsilon z)$ where $\alpha\delta - \beta\gamma = 1$; find the matrix of $f^*: \Lambda^2\mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ associated to basis $dy\wedge dz$, $dz\wedge dx$, $dx\wedge dy$.

Consider the value of f^* on the basis elements:

$$f^*(dy\wedge dz) = f^*dy \wedge f^*dz = d(y\circ f)\wedge d(z\circ f)$$
$$= d(\gamma x + \delta y)\wedge d(\varepsilon z)$$
$$= (\gamma dx + \delta dy)\wedge \varepsilon dz$$
$$= \gamma\varepsilon\, dx\wedge dz + \delta\varepsilon\, dy\wedge dz$$
$$= \underline{\delta\varepsilon\, dy\wedge dz - \gamma\varepsilon\, dz\wedge dx}.$$

$$\Rightarrow f^*\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\delta\varepsilon\\-\gamma\varepsilon\\0\end{pmatrix}$$

$$f^*(dz\wedge dx) = f^*dz\wedge f^*dx = d(z\circ f)\wedge d(x\circ f)$$
$$= d(\varepsilon z)\wedge d(\alpha x + \beta y)$$
$$= \varepsilon\alpha\, dz\wedge dx + \varepsilon\beta\, dz\wedge dy$$
$$= \varepsilon\alpha\, dz\wedge dx - \varepsilon\beta\, dy\wedge dz$$

$$\Rightarrow f^*\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\varepsilon\beta\\\varepsilon\alpha\\0\end{pmatrix}$$

$$f^*(dx\wedge dy) = f^*dx\wedge f^*dy = d(x\circ f)\wedge d(y\circ f)$$
$$= d(\alpha x + \beta y)\wedge d(\gamma x + \delta y)$$
$$= (\alpha dx + \beta dy)\wedge(\gamma dx + \delta dy)$$
$$= \alpha\gamma\, \underbrace{dx\wedge dx}_{0} + \alpha\delta\, dx\wedge dy + \beta\gamma\, dy\wedge dx + \beta\delta\, \underbrace{dy\wedge dy}_{0}$$
$$= (\alpha\delta - \beta\gamma)\, dx\wedge dy = dx\wedge dy$$

$$\Rightarrow f^*\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

So
$$f^* = \begin{bmatrix}\delta\varepsilon & -\varepsilon\beta & 0\\ -\gamma\varepsilon & \varepsilon\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

④ (a) $\pi_1(\mathbb{R}P^2 \setminus \{pt\})$

(b) Show that any map $f: \mathbb{R}P^2 \to \mathbb{R}P^2$ which is not surjective is homotopic to constant map

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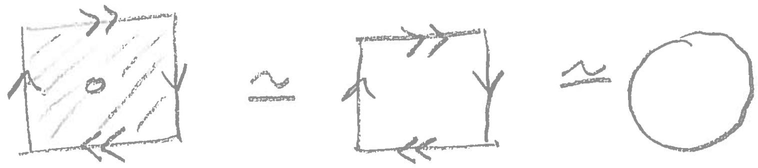
(a)  , i.e. $\mathbb{R}P^2 \setminus \{pt\}$

deformation retracts to circle, hence:

$$\pi_1(\mathbb{R}P^2 \setminus \{pt\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

(b) Suppose that f is not surjective, i.e. suppose $\operatorname{im} f = \mathbb{R}P^2 \setminus \{pt\} \simeq S^1$; hence we have the induced map $f_*: \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2 \setminus pt) \cong \pi_1(S^1)$

$\Rightarrow f_*: \mathbb{Z}_2 \to \mathbb{Z} \Rightarrow f_* \equiv 0$ since $\mathbb{Z}$ has no finite-order elements

Hence for covering space $p: \mathbb{R} \to S^1$, we have $f_*(\pi_1(\mathbb{R}P^2)) = 1 \leq p_*(\pi_1(\mathbb{R})) = 1$, hence we have a lift:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{R}P^2 \setminus \{pt\} \end{tikzcd}$$

Now $\mathbb{R}$ is contractible, hence $\exists$ homotopy $h_t: \mathbb{R} \to \mathbb{R}$ s.t. $h_0 = id_{\mathbb{R}}$, $h_1 = $ const.; then see that $h_0 \circ \tilde{f} = \tilde{f}$ and $h_1 \circ \tilde{f} = $ const., hence $h_t \circ \tilde{f} := g_t$ is homotopy of $\tilde{f}$ to const, hence, since $f = p \circ \tilde{f}$, we have $p \circ g_0 = p \circ \tilde{f} = f$ and $p \circ g_1 = $ const, hence $p \circ g_t$ is homotopy of f to const, i.e. $f \simeq$ const

5. B^2, $\partial B^2 = S^1$, $X = S^1 \times B^2$, $\partial X = S^1 \times S^1$

Compute $H_n(X, \partial X)$:

Consider the Long Exact Sequence of Relative Homology:

$$\cdots \to 0 \to H_3(X,\partial X) \to H_2(\partial X) \to H_2(X) \to H_2(X,\partial X)$$
$$\to H_1(\partial X) \xrightarrow{i^*} H_1(X) \to H_1(X,\partial X) \to H_0(\partial X) \to H_0(X)$$
$$\to H_0(X,\partial X) \to 0$$

Recall that $X = S^1 \times B^2 \simeq$ solid $T^2 \simeq S^1$

& $\partial X = S^1 \times S^1 \cong T^2$.

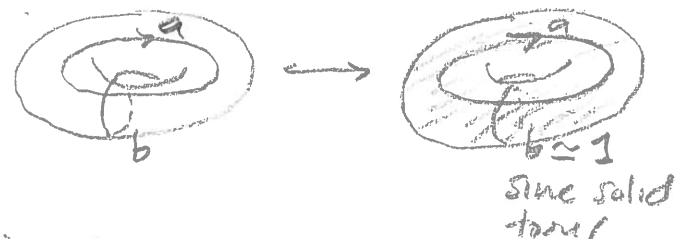
So: $H_i(X) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$ and $H_i(\partial X) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$

and then:

$$0 \to H_3(X,\partial X) \to \mathbb{Z} \to 0 \to H_2(X,\partial X) \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i^*} \mathbb{Z}$$
$$\to H_1(X,\partial X) \to \mathbb{Z} \xrightarrow{i_0^*} \mathbb{Z} \to H_0(X,\partial X) \to 0$$

Clearly $H_3(X,\partial X) \cong \mathbb{Z}$ by exactness.

Now consider the inclusion map $i : \partial X \hookrightarrow X$ and the induced homomorphism $i^* : H_1(\partial X) \cong \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \to H_1(X) \cong \mathbb{Z}\langle a\rangle$



hence $i^*(a) = a$

$i^*(b) = 0$, hence $\ker i^* = \mathbb{Z}$

and $\operatorname{im} i^* = \mathbb{Z}$.

Furthermore, the induced hom

$i^* : H_0(\partial X) \to H_0(X)$ is injective

since i is inclusion & both path-connected.

$\Rightarrow \ker i^* = 0$ here.

Now consider the subsequence:

$$0 \longrightarrow H_2(X,\partial X) \xrightarrow{\alpha} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i^*} \mathbb{Z} \xrightarrow{\beta} H_1(X,\partial X) \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{i_0^*} \mathbb{Z} \xrightarrow{\delta} H_0(X,\partial X) \to 0$$

Now, $\ker i^* = \mathbb{Z} \Rightarrow \underline{\mathbb{Z} = \operatorname{im}\alpha \cong H_2(X,\partial X)}$ since α inj. by exactness

We also have $\ker i_0^* = 0 \Rightarrow \operatorname{im} i_0^* = \mathbb{Z} \Rightarrow \ker\delta = \mathbb{Z}$

$\Rightarrow \operatorname{im}\delta = 0$, but δ surjective by exactness, hence $\underline{H_0(X,\partial X) = 0}$

On the other hand, $\ker i_0^* = 0 \Rightarrow \operatorname{im}\gamma = 0 \Rightarrow$

$\ker\gamma = H_1(X,\partial X)$, but $\operatorname{im} i^* = \mathbb{Z} \Rightarrow \ker\beta \doteq \mathbb{Z}$

$\Rightarrow \operatorname{im}\beta = 0 \Rightarrow \ker\gamma = 0 \Rightarrow \underline{H_1(X,\partial X) = 0}$

So we have $\boxed{H_i(X,\partial X) \cong \begin{cases} 0 & i=0,1 \\ \mathbb{Z} & i=2,3 \\ 0 & i>3. \end{cases}}$

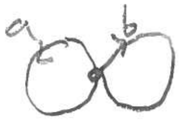
(6.) $X = S^1 \vee S^1$, $p : \tilde{X} \to X$ covering space s.t. $|p^{-1}(x)| = 2$

Compute $\pi_1(\tilde{X})$.

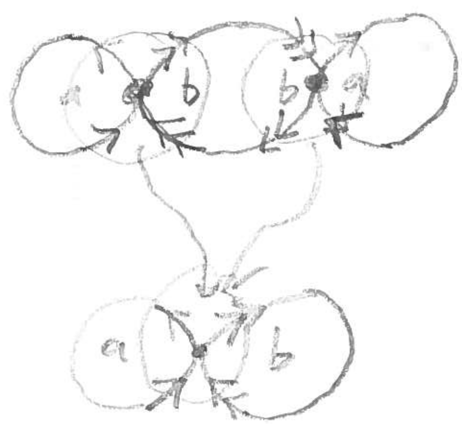
#1: By Seifert-van Kampen, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$, the free group on 2 generators.

Then a 2-sheeted connected cover corresponds to an index 2 sg. $H \leq F_2$, i.e. $H = F_3$ $\left(H \leq F_{n+1},\ [F_{n+1} : H] = k \Rightarrow H = F_{kn+1}\right)$

So $\pi_1(\tilde{X}) = F_3$

#2: 

$= S^1 \vee S^1$; a 2-sheeted cover will be compact (X cpt, $\tilde{X}$ finite sheeted $\Rightarrow$ $\tilde{X}$ cpt) and have 2 wedge points:



So $\tilde{X} =$ 

& $\pi_1(\tilde{X}) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = F_3$

(7.) What are the cpt, connected surfaces S for which $\exists$ $f: S \to S$ immersion that is not a diffeomorphism?

Since $f: S \to S$ is immersion, $T_p f: T_p S \cong \mathbb{R}^2 \to T_{f(p)} S \cong \mathbb{R}^2$ is injective for all p, hence invertible by dimension, hence local diffeo. at p by Inverse Fn. Thm, hence local diffeo at all $p \in M$, hence f a local diffeo.

- A local diffeo. is an open/closed map, hence since S is open/closed, $f(S)$ is open/closed, hence $f(S) = S$ by connectivity of S

- For $p \in S$, $f^{-1}(p)$ is closed subset of cpt S, hence cpt, hence finite # of points, i.e. $f^{-1}(p) = \{q_1, \dots, q_k\}$.

- Now, choose neighborhoods $U_{q_1}, \dots, U_{q_k}$ s.t. f is diffeo at $q_1, \dots, q_k$, hence $p \in f(U_{q_i}) \approx U_{q_i}$

Now consider the nbhd:

$$p \in V = f(U_{q_1}) \cap \dots \cap f(U_{q_k})$$

This is clearly evenly-covered by disjoint nbhds $V_{q_i} \subseteq U_{q_i}$ (since restriction of homeomorphism is homeo. onto image)

$\Rightarrow$ Therefore, f is a covering map, hence for f not diffeomorphism, f will be n-sheeted cover for $n > 1$, and then; since $\chi(\tilde{X}) = n\chi(X)$, we have:

$\chi(S) = n\chi(S) \Rightarrow \chi(S) = 0$ since $n > 1$

Then $0 = \chi(M_g) = 2 - 2g \Rightarrow g = 1 \Rightarrow S = \text{Torus}$
or $0 = \chi(N_g) = 2 - g \Rightarrow g = 2 \Rightarrow S = \text{Klein bottle}$