Graduate Exam in Topology/Geometry

February 2002

1. Let $P^n(\mathbb{R})$ be the projective n–space, namely the quotient space of the sphere S^n by the equivalence relation $\sim$ defined by $x \sim y \Leftrightarrow x = \pm y$.

(a) Show that $P^n(\mathbb{R})$ is a manifold.

(b) Show that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

2. In the set $M(n)$ of all $n \times n$ matrices, identified to $\mathbb{R}^{n^2}$, consider the subset $O(n)$ consisting of the orthogonal matrices, namely those matrices A for which AA^t is the identity (where A^t denotes the transpose). Show that $O(n)$ is a submanifold of $M(n) = \mathbb{R}^{n^2}$, and that the tangent space $T_{\mathrm{Id}}O(n)$ at the identity Id is equal to the space of all antisymmetric matrices (namely those matrices for which $A^t = -A$).

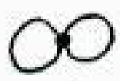
3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (\alpha x + \beta y, \gamma x + \delta y, \varepsilon z)$, where α, β, γ, δ, ε are constants with $\alpha\delta - \beta\gamma = 1$. Find the matrix of $f^* : \wedge^2\mathbb{R}^3 \to \wedge^2\mathbb{R}^3$ associated to the basis $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$.

4. Let $P^2(\mathbb{R})$ be the real projective plane.

(a) If $x \in P^2(\mathbb{R})$, compute the fundamental group $\pi_1(P^2(\mathbb{R}) - \{x\})$.

(b) Show that any map $f : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which is not surjective is homotopic to a constant map. (Hint: use a covering space).

5. Let B^2 be the closed 2–dimensional ball, with boundary the circle S^1. Let $X = S^1 \times B^2$ and let $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_n(X, \partial X)$ with coefficients in Z. (You are allowed to use whatever you may know about the homology of the torus ∂X).

6. Let X be the figure eight  , union of two circles C_1 and C_2 meeting in one point. Let $p : \widetilde{X} \to X$ be a covering space such that $\widetilde{X}$ is connected and such that the preimage $p^{-1}(x)$ of each $x \in X$ consists of 2 points. Compute the fundamental group of $\widetilde{X}$.

7. What are the compact connected surfaces S for which there exists an immersion $S \to S$ which is not a diffeomorphism? (Hint: Euler characteristic).