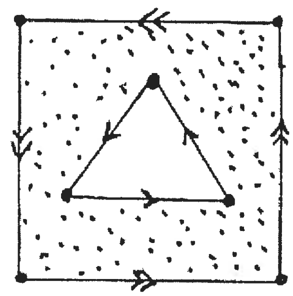
Geometry/Topology Graduate Exam
Fall 1999

1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



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2. Let X be the space described in 1. Compute the homology groups $H_n(X;\mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.

3. Give an example of a path connected space X which admits no covering $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected.

4. Let X be a path connected manifold with $\pi_1(X;x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

5. You may know that there exist continuous surjective maps $f : [0,1] \to [0,1]^2$ from the interval onto the square. Show that there exists no *continuously differentiable* surjective map $f : [0,1] \to [0,1]^2$.

6. Consider the map $\varphi : S^1 \times S^1 \to S^1 \times S^1$ defined by $\varphi(u,v) = (u^5, v^{-3})$, where we identify S^1 to the unit circle in the complex plane $\mathbb{C}$. Compute the degree of φ.

7. Let $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ be a closed (namely $d\omega = 0$) differential form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Consider the homomorphism $i^* : \Omega^n(\mathbb{R}^{n+1} - \{0\}) \to \Omega^n(S^n)$ induced by the inclusion map $i : S^n \to \mathbb{R}^{n+1} - \{0\}$. Show that the form ω is exact (namely there exists $\alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ such that $\omega = d\alpha$) if and only if $\int_{S^n} i^*(\omega) = 0$.

1. Van Kampen
2. Cellular Homology
3. Shrinking Wedge of circles (not semilocally simply connected, $\forall$ U around basept, $\pi_1(U) \to \pi_1(X)$ not trivial)
4. Classification of covering spaces
5. Corollary to Sard's Theorem.
6. Each point has 15 preimages, each of which is orientation reversing.
7. Stokes, $H^n_{dR}(S^n) \cong \mathbb{R}$ via I.

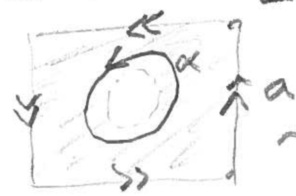
Geo/Top Fall '99:

(1.) $X =$  ; $\pi_1(X)$.

Let $A =$  , $B =$  , $A \cap B \simeq$  $= S^1$; so $\underline{A \cup B = X}$

Now consider the inclusions; see that $A \simeq S^1$ & $B \simeq S^1$, so:

$$i_1^*: \underset{\mathbb{Z}\langle\alpha\rangle}{\pi_1(A\cap B)} \longrightarrow \underset{\mathbb{Z}\langle a\rangle}{\pi_1(A)} \qquad i_2^*: \underset{\mathbb{Z}\langle\alpha\rangle}{\pi_1(A\cap B)} \longrightarrow \underset{\mathbb{Z}\langle b\rangle}{\pi_1(B)}$$



$\rightsquigarrow i_1^*(\alpha) = a^4$

$\rightsquigarrow i_2^*(\alpha) = b^3$

So apply Seifert-van Kampen:

$$\pi_1(X) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1^*(\alpha)\, i_2^*(\alpha)^{-1} \rangle$$

$$= \mathbb{Z}\langle a\rangle * \mathbb{Z}\langle b\rangle \Big/ \langle a^4 b^{-3}\rangle = \underline{\langle a, b : a^4 b^{-3} = 1\rangle}$$

(2.) $\underline{H_i(X) \text{ for same } X}$:

Apply Mayer-Vietoris with the same decomposition:

$$0 \longrightarrow H_2(X) \longrightarrow H_1(A\cap B) \longrightarrow H_1(A)\oplus H_1(B) \longrightarrow H_1(X)$$

$$\longrightarrow H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(X) \longrightarrow 0$$

$$H_i(A) \cong H_i(B) \cong H_i(A\cap B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & i > 1 \end{cases}$$

$\underline{H_0(X) \cong \mathbb{Z}}$ since X path-connected

So now we have:

$$0 \to H_2(X) \xrightarrow{\psi} \mathbb{Z}\langle\alpha\rangle \xrightarrow{(i^*,j^*)} \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \xrightarrow{\varphi} H_1(X) \xrightarrow{\alpha} \mathbb{Z}$$
$$\xrightarrow{\beta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \longrightarrow 0$$

- Consider the inclusions again:

$$i^*: \underset{\mathbb{Z}\langle\alpha\rangle}{\underset{\parallel}{H_1(A\cap B)}} \to \underset{\mathbb{Z}\langle a\rangle}{\underset{\parallel}{H_1(A)}} \qquad j^*: H_1(A\cap B) \longrightarrow H_1(B)$$
$$\mathbb{Z}\langle\alpha\rangle \longrightarrow \mathbb{Z}\langle b\rangle$$

$$i^*(\alpha) = a^4 \qquad j^*(\alpha) = b^3$$

same reasoning as for van kampen

So $\operatorname{im}(i^*,j^*) = \langle(a^4, b^3)\rangle \cong \mathbb{Z} \Rightarrow (i^*,j^*)$ injective

$\Rightarrow \ker\varphi = \langle(a^4, b^3)\rangle \cong$

- Now, γ is surjective, hence $\ker\gamma = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = 0$

$\Rightarrow \operatorname{im}\alpha = 0 \Rightarrow \varphi$ surjective.

Hence by 1st iso theorem, $H_1(X) \cong \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle / \langle a^4, b^3\rangle$

$\cong \underline{\mathbb{Z}_4 \oplus \mathbb{Z}_3}$.

- Now, ψ must be injective, hence $\operatorname{im}\psi = H_2(X)$,

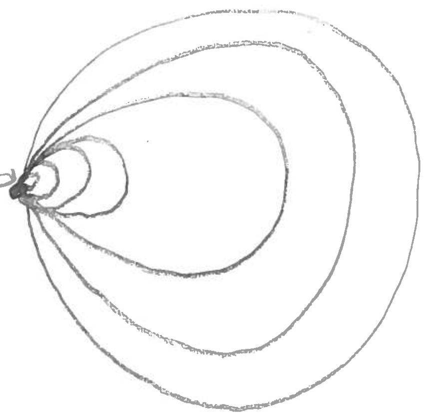
and $\operatorname{im}\psi = \ker(i^*,j^*) = 0$, so $H_2(X) \cong 0$

So we have: $H_i(X) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_4\oplus\mathbb{Z}_3 & i=1 \\ 0 & i>1 \end{cases}$

③ Example of path-connected X with no universal cover:

A space has a universal cover iff it is semi-locally simply-connected, i.e. each point $p \in X$ has a neighborhood U for which the inclusion $\pi_1(U,p) \hookrightarrow \pi_1(X,p)$ is trivial.

Now consider the shrinking wedge of circles:



x_0

$= X$

Any neighborhood of x_0 will contain a circle, hence the inclusion $\pi_1(U,x_0) \hookrightarrow \pi_1(X,x_0)$ may never be trivial, hence X has no universal cover.

(4.) X path-connected, $\pi_1(X, x_0) = \mathbb{Z}_5$, $p: \tilde{X} \to X$ covering space with $|p^{-1}(x_0)| = 6$. Show that $\tilde{X}$ has either 2 or 6 connected components

First, $\tilde{X}$ cannot be connected since $\mathbb{Z}_5$ has no index 6 subgroup (by classification of connected covering spaces).

Now, recall that a covering space can be given by the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, i.e. 6-sheeted covering spaces are classified by the homomorphisms

$$f: \pi_1(X, x_0) \longrightarrow S_6 \rightsquigarrow f: \mathbb{Z}_5\langle a \rangle \longrightarrow S_6.$$

So $f(a)$ must have order 5, hence it is id or a 5-cycle. Recall that these permutations are given by lifts of loops in X to paths between the members of fiber $f^{-1}(x_0) := \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_6\}$.

- Case 5-cycle: A 5-cycle permutes 5 $\tilde{x}_i$ and fixes one, hence 5 of them must be in a path component. This leaves the last one to be in its own component (not connected!), hence 2 components

- Case id: every pt $\tilde{x}_i$ is fixed, hence there must only be a loop based at each in $\tilde{X}$, hence each resides in its own path component, i.e. 6 components

(5.) $\exists$ cont, surjective $f:[0,1]\to[0,1]^2$ (space-filling curve)
Show $\nexists$ cont, differentiable, surjective $f:[0,1]\to[0,1]\times[0,1]$

Suppose $f:[0,1]\to[0,1]\times[0,1]$ is surjective + differentiable.
Now consider the tangent map: $T_pf: T_pI\to T_pI^2$,
i.e. $T_pf:\mathbb{R}\to\mathbb{R}^2$, hence T_pf may never be surjective,
hence f has no regular pts, hence f has no
regular values, hence $\operatorname{im} f$ meas. 0 by Sard.
But $I\times I$ is not measure 0, hence f not
surjective.

(6.) $\varphi: S^1\times S^1\to S^1\times S^1$, $\varphi(z,w)=(z^5,w^{-3})$ $(S^1\subseteq\mathbb{C})$.
Deg φ = ?

We have the volume form $d\theta_1\wedge d\theta_2$ on the torus,
so consider the induced map on top-dim'l de Rham.

$\varphi^*: H^2(S^1\times S^1)\longrightarrow H^2(S^1\times S^1)$

$$\begin{aligned}\varphi^*(d\theta_1\wedge d\theta_2) &= \varphi^*d\theta_1\wedge\varphi^*d\theta_2\\ &= d(\theta_1\circ\varphi)\wedge d(\theta_2\circ\varphi)\\ &= d(5\theta_1)\wedge d(-3\theta_2)\\ &= -15\,d\theta_1\wedge d\theta_2 \Rightarrow \underline{\deg\varphi = -15}\end{aligned}$$

$\varphi(z,w)=\varphi(e^{i\theta_1},e^{i\theta_2})=(e^{i5\theta_1},e^{-i3\theta_2})$

7. $\omega \in \Omega^n(\mathbb{R}^{n+1} \setminus \{0\})$ closed

Consider $i^* : \Omega^n(\mathbb{R}^{n+1} \setminus \{0\}) \longrightarrow \Omega^n(S^n)$

Show ω exact $\iff \int_{S^n} i^*(\omega) = 0$

($\Rightarrow$) Suppose ω exact. Then $[\omega] = 0 \in H^n(\mathbb{R}^{n+1} \setminus \{0\})$,

and $H^n(\mathbb{R}^{n+1} \setminus \{0\}) \cong H^n(S^n)$, hence $[i^*(\omega)] = 0 \in H^n(S^n)$

now apply top-dimensional de Rham isomorphism:

$$I : H^n(S^n) \longrightarrow \mathbb{R}$$
$$\underset{\overset{\|}{0}}{[i^*(\omega)]} \longmapsto \underline{\underline{\int_{S^n} i^*(\omega) = 0}}$$

($\Leftarrow$) Suppose $\int_{S^n} i^*(\omega) = 0$

By the top-dim. de Rham iso, this means $[i^*(\omega)] = 0 \in H^n(S^n)$,

but again $H^n(S^n) \cong H^n(\mathbb{R}^{n+1} \setminus \{0\})$,

hence $[\omega] = 0$ since $[i^*(\omega)] = i^*([\omega]) = 0$

and i^* is isomorphism.