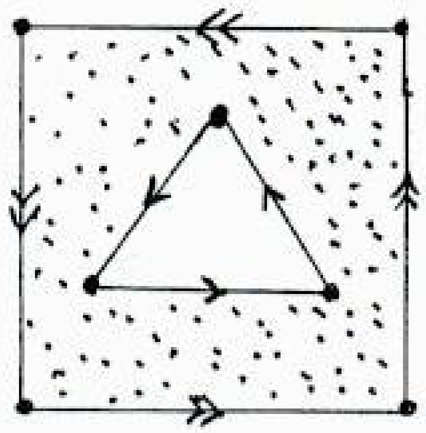
Geometry/Topology Graduate Exam
Fall 1999

1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



2. Let X be the space described in 1. Compute the homology groups $H_n(X; \mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.

3. Give an example of a path connected space X which admits no covering $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected.

4. Let X be a path connected manifold with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

5. You may know that there exist continuous surjective maps $f : [0,1] \to [0,1]^2$ from the interval onto the square. Show that there exists no *continuously differentiable* surjective map $f : [0,1] \to [0,1]^2$.

6. Consider the map $\varphi : S^1 \times S^1 \to S^1 \times S^1$ defined by $\varphi(u,v) = (u^5, v^{-3})$, where we identify S^1 to the unit circle in the complex plane $\mathbb{C}$. Compute the degree of φ.

7. Let $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ be a closed (namely $d\omega = 0$) differential form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Consider the homomorphism $i^* : \Omega^n(\mathbb{R}^{n+1} - \{0\}) \to \Omega^n(S^n)$ induced by the inclusion map $i : S^n \to \mathbb{R}^{n+1} - \{0\}$. Show that the form ω is exact (namely there exists $\alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ such that $\omega = d\alpha$) if and only if $\int_{S^n} i^*(\omega) = 0$.