GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

FALL 1994

Problem 1 Let $X = \mathbb{R}^2 - \{(\frac{1}{n}, 0) | n = 1, 2, \ldots\}$.

(a) Show that the fundamental group $\pi_1(X, (0,0))$ is non-trivial.

(b) Is the fundamental group abelian? Explain.

(c) Is X semi-locally simply connected? Explain.

(d) Does there exist a covering $E \to X$ with E simply connected?

Problem 2 Let M be a manifold of dimension $m \geq 2$ and let $B \subset M$ be an open subset that is homeomorphic to the m-dimensional open ball. Fix $x \in B$ and consider the homoemorphisms

$$H_m(M) \xrightarrow{\alpha} H_m(M, M - \{x\}) \xrightarrow{\beta} H_m(b, B - \{x\}) \xrightarrow{\gamma} H_{m-1}(B - \{x\})$$

where α is induced by the inclusion map $M \to (M, M - \{x\})$, β is the excision isomorphism, and γ is the connecting homomorphism of the long exact sequence in relative homology of the pair $(B, B - \{x\})$. Also, let $H_m(M) \xrightarrow{\delta} H_{m-1}(B - \{x\})$ be the connecting homomorphism of the Mayer-Vietoris exact sequence associated to the decomposition of M as $M = (M - \{x\}) \cup B$. Is δ equal to the composition $\gamma \circ \beta \circ \alpha$?

Problem 3 Let $\boldsymbol{S}^3 \subset \mathbb{R}^4$ be the 3-sphere defined by $w^2 + x^2 + y^2 + z^2 = 1$ where w, x, y, z are the standard Euclidean coordinates on $\mathbb{R}^4$. Let $f : \boldsymbol{S}^3 \hookrightarrow \mathbb{R}^4$ be the inclusion map. Compute the integral of $f * \theta$ over $\boldsymbol{S}^3$, where θ is the 3-form (defined on $\mathbb{R}^4$ minus the origin) given by

$$\theta = \frac{w^7 \, \mathrm{d}x \wedge \, \mathrm{d}y \wedge \, \mathrm{d}z}{w^2 + z^2 + y^2 + z^2}$$

Problem 4 Is the set $X \subset \mathbb{R}^4$ defined by $w^2 + x^2 + y^2 + z^2 = 1$ and $w^2 + x^2 = y^2 + z^2$ a smooth submanifold of $\mathbb{R}^4$?

Problem 5 Can the set $X \subset \mathbb{R}^4$ defined by $w^2 + x^2 + y^2 + z^2 < 1$ and $w^2 + x^2 = y^2 + z^2$ (considered as a topological subspace of $\mathbb{R}^4$) carry the structure of a smooth manifold?

Problem 6 Let $\boldsymbol{S}^n$ be the n-dimensional sphere, and let $\boldsymbol{T}^n = (\boldsymbol{S}^1)^n$ be the n-dimensional torus. Does there exist a submersion from $\boldsymbol{S}^3$ to $\boldsymbol{T}^2$? From $\boldsymbol{T}^2$ to $\boldsymbol{S}^2$? From $\boldsymbol{S}^3$ to $\boldsymbol{S}^2$? (Note: A submersion is a smooth map whose differential at each point is surjective.)