Topology Qualifying Exam Fall 2025

1. Consider S^3, viewed as the one point compactification of $\mathbb{R}^3$ with the point $\star$ at infinity. Let $X \subset S^3$ be the subspace given by the following union.

$$\{(x,y,z) \,:\, x^2+y^2=1\} \cup \{(x,y,z) \,:\, |z|=1 \text{ and } x^2+y^2 \leqslant 1\} \cup \{\star\}$$

   (a) Describe a cell decomposition of X and draw a picture of it.

   (b) Compute the cellular homology of X using the cell decomposition in (a).

   (c) Is X homotopy equivalent to a wedge sum of m copies of the k-sphere for some m and k? Either describe a homotopy equivalence or prove that one does not exist. Your answer should be 2-3 sentences.

   Next, consider a space W admitting an open cover $W = A \cup B$ such that

$$H_4(A\cap B) \cong H_3(A) \cong H_3(B) \cong 0,$$
$$H_3(A\cap B) \cong H_4(A) \cong H_4(B) \cong \mathbb{Z}/2\mathbb{Z}.$$

   (d) Prove that W does not admit a cell decomposition with only one 4-cell.

2. Let $X = \mathbb{RP}^2 \vee S^1$.

   (a) Compute $\pi_1(X)$.

   (b) Show that X has a connected 3-sheeted cover which is regular.

   (c) Compute the group of deck transformations of the cover from part (b).

   (d) Show that X has a connected 3-sheeted cover which is not regular.

3. Consider the space $Y = (\mathbb{RP}^3 \times \mathbb{T}^2) \vee \mathbb{R}/\mathbb{Z}$.

   (a) Compute the singular homology groups (with $\mathbb{Z}$ coefficients) of Y.

   (b) Is Y homotopy equivalent to a orientable closed manifold?

   (c) Compute the fundamental group $\pi_1(Y)$.

   Next, fix $p \in \mathbb{RP}^3$ and consider the loops $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{RP}^3 \times \mathbb{T}^2$ and $\eta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by

$$\gamma(t) = (p,t,0) \in \mathbb{RP}^3 \times (\mathbb{R}/\mathbb{Z})^2 = \mathbb{RP}^3 \times \mathbb{T}^2 \qquad \text{and} \qquad \eta(t) = t$$

   Since $\mathbb{RP}^3 \times \mathbb{T}^2$ and $\mathbb{R}/\mathbb{Z}$ naturally include into Y, we may view γ and η as loops in Y.

   (d) Is there a homotopy equivalence $F : Y \to Y$ such that $F \circ \gamma$ is homotopic to η? Prove or disprove.

4. Let $U(n) \subset GL(n,\mathbb{C})$ be the group of unitary $n \times n$ matrices A. Let e_1 be the unit vector $(1,0,\ldots,0)$ in the unit sphere S^{2n-1} of $\mathbb{C}^n$ and consider the map

$$\pi : U(n) \to S^{2n-1} \qquad \text{given by} \qquad \pi(A) = Ae_1 \in S^{2n-1}$$

   Note that π is a fibration (you do not need to prove this).

   (a) Compute the fiber of π at e_1.

   (b) Compute the homotopy groups $\pi_1(U(n))$ and $\pi_2(U(n))$ for all $n \geqslant 1$.