Fall 2023 Solutions

Problem 3

Let X be a manifold with $\pi_2(X,x)=0$ for all $x\in X$. Is it necessarily the case that $H_2(X)=0$ as well?

proof:
The torus T^2 is a manifold.

We know from

> Torus Homology Groups
>
> # Homology
>
> $$H_n(T^2)=\begin{cases}\mathbb{Z}, & \text{if } n=0,2\\ \mathbb{Z}^2, & \text{if } n=1\\ 0, & \text{otherwise}\end{cases}$$
>
> # Reduced Homology
>
> $$\tilde{H}_n(T^2)=\begin{cases}\mathbb{Z}, & \text{if } n=2\\ \mathbb{Z}^2, & \text{if } n=1\\ 0, & \text{otherwise}\end{cases}$$
>
> #Algebraic_Topology

$$H_n(T^2)=\begin{cases}\mathbb{Z}, & \text{if } n=0,2\\ \mathbb{Z}^2, & \text{if } n=1\\ 0, & \text{otherwise}\end{cases}$$

By

> Covering Space Homotopy Group Isomorphism
>
> **Rotman Theorem 11.29**
> If $(\tilde{X},p)$ is a covering space of X, then
>
> $$p_*:\pi_n(\tilde{X})\to\pi_n(X)$$
>
> is an isomorphism for all $n\geq 2$.
>
> In other words $\pi_n(\tilde{X})\cong\pi_n(X)$ for all $n\geq 2$.
>
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we have that if $(\tilde{X},p)$ is a covering space of X then

$$\pi_n(\tilde{X})\cong\pi_n(X)\quad\text{for } n\geq 2$$

We know that $T^2=S^1\times S^1$
$\mathbb{R}$ is a universal cover of S^1
Thus $S^1\times S^1$ has $\mathbb{R}^2$ as a universal cover

By

> Contractible Space Has Trivial Homotopy Groups
>
> **Rotman 11.28**
>
> If X is contractible, then $\pi_n(X,x_0)=0$ for all $n\geq 0$.
>
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, since $\mathbb{R}^2$ is contractible, $\pi_n(\mathbb{R}^2) = 0$ for all $n \geq 2$

Thus $\pi_n(T^2) \cong \pi_n(\mathbb{R}^2) = 0$ for all $n \geq 2$ and we have that $\pi_2(T^2, x) = 0$ for all $x \in T^2$, but $H_2(T^2; \mathbb{Z}) = \mathbb{Z}$.

■

Approach 2 (Direct Sum Isomorphism)

proof:
The torus T^2 is a manifold.

We know that

$$H_n(T^2) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2 \\ \mathbb{Z}^2, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $T^2 = S^1 \times S^1$ then we have by

> Homotopy Groups of Cartesian Product of Spaces isomorphic to Direct Product of Homotopy Groups of Spaces
>
> **Rotman Exercise 11.24**
>
> If X and Y are pointed spaces, then, for all $n \geq 2$,
>
> $$\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y)$$
>
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, that

$$\pi_n(T^2) \cong \pi_n(S^1 \times S^1) \cong \pi_n(S^1) \oplus \pi_n(S^1) \cong 0 \quad \text{for } n > 1$$

Thus $H_2(T^2; \mathbb{Z}) \cong \mathbb{Z}$ when $\pi_2(T^2) \cong 0$. ■

Problem 4

What are the integral homology groups of $S^1 \vee S^2 \vee S^3 \vee S^4$

By

> Reduced Homology of Wedge Sum
>
> **Hatcher Ch.2 Ex. 31**
> If the basepoints of X and Y that are identified in $X \vee Y$ are Deformation Retracts of neighborhoods $U \subset X$ and $V \subset Y$ then
>
> $$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$
>
> **Julian Take-home:**
> if (X, x) and (Y, y) are both good pairs, then
>
> $$\tilde{H}_n(X \vee Y) \simeq \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$
>
> ## Proof
>
> Can be shown by mayer-vietoris
>
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we have

$$H_n(S^1 \vee S^2 \vee S^3 \vee S^4) \cong \begin{cases} \mathbb{Z} & n = 0, 1, 2, 3, 4 \\ 0 & \text{else} \end{cases}$$

Problem 6

Give an example of a

> Covering Space
>
> **Rotman: p.273**
> If X is a topological space, then an ordered pair $(\tilde{X}, p)$ is a *covering space* of X if:
>
> 1) $\tilde{X}$ is a path connected topological space
> 2) $p : \tilde{X} \to X$ is continuous
> 3) each $x \in X$ has an open neighborhood $U = U_x$ that is evenly covered by p.
>
> **Hatcher: p.56**
> A *covering space* of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p. Such a U is called Evenly Covered and the disjoint open sets in $\tilde{X}$ that project homeomorphically to U by p are called Sheets of $\tilde{X}$ over U.
>
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$X \to Y$ which is not a

> Regular Covering Space
>
> **Rotman p.283**
> A covering space $(\tilde{X}, p)$ of X is *regular* if $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$ for every $x_0 \in X$
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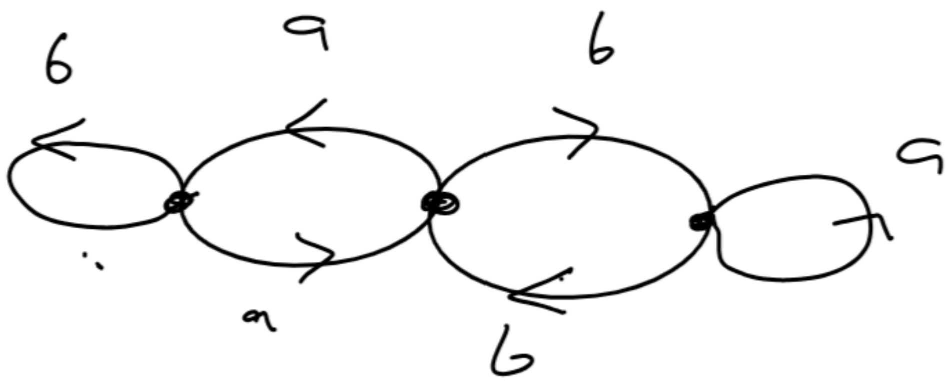
proof:

Take $Y = S^1 \vee S^1$ with fundamental group $\langle a, b \rangle$ and let the covering space X be the picture below

!



Which has fundamental group $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$. Then it is not a regular covering space, since this subgroup does not contain the element

$$a\, aba^{-1}\, a^{-1} = a^2ba^{-2}$$

■

Problem 7

Show that the Cantor set does not admit a CW complex structure

Don't think this would show up on an exam.

Cantor sect is compact with infinitely many connected components, compact CW spaces can only have finitely many.

From stack exchange:

Mariano: This is still a good example for the homotopy-equivalence question: If $f : C \to X$ is a h.e. from the Cantor set to a CW-complex, then $f(C)$ has to have nonempty intersection with each component of X and there has to be continuum of such components. Since each component of X is open, we obtain an open covering of C by continuum of disjoint nonempty open sets, which is, of course, impossible. – Moishe Kohan Oct 12, 2013 at 11:49