2017, Fall

Problem 1.

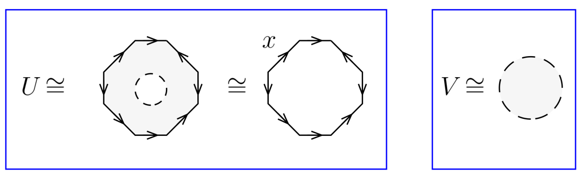
Since M is compact and f is continuous, $\mathsf{im}(f)$ is compact, and in particular not all of $\mathbb{R}^m$. So f isn't surjective, and $\mathsf{deg}(f) = 0$. Let $y \in \mathbb{R}^m$ be a regular value of f; by Sard, such points have full measure in $\mathbb{R}^m$. We have

$$0 = \mathsf{deg}(f) = \sum_{x \in f^{-1}(y)} \mathsf{deg}_x(f).$$

But each local degree $\mathsf{deg}_x(f) = \pm 1$, so to obtain 0 on the left-hand side, there must be an even number of points belonging to $f^{-1}(y)$. $\square$

Problem 2.

Let X be the given quotient space, and write X as the union of the subspaces U and V shown below, with $U \cap V \cong \mathsf{S}^1$.



Let $x \in \pi_1(U)$ correspond to the edge above as labeled. Observe that $\pi_1(V) \cong 1$ since V is contractible, and that $\pi_1(U) \cong \mathbb{Z}$, generated by the single element x. Letting $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions, then the induced homomorphism $i_* : \pi_1(U \cap V) \to \pi_1(U)$ maps the single generator $1 \in \pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ to $i_*(1) = xxx^{-1}xx^{-1}x^{-1}x^{-1}x^{-1} = x^{-2}$, and $j_* : \pi_1(U \cap V) \to \pi_1(V)$ maps it to $j_*(1) = 1$ by triviality of $\pi_1(V)$. So by van Kampen,

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{\langle x \rangle}{\langle x^{-2} \rangle} = \langle x \mid x^{-2} = 1 \rangle = \langle x \mid x^2 = 1 \rangle \cong \mathbb{Z}_2.$$

$\square$

Problem 3.

- Letting $\mathsf{Bij}(p^{-1}(x_0))$ denote the set of bijections $p^{-1}(x_0) \to p^{-1}(x_0)$, we have an assignment

  $$F : \pi_1(X, x_0) \to \mathsf{Bij}(p^{-1}(x_0)), \quad F_{[\gamma]}(\tilde{x}) := \tilde{\gamma}_{\tilde{x}}(1),$$

  where $\tilde{\gamma}_{\tilde{x}} : [0,1] \to \tilde{X}$ is the unique lift of γ satisfying $\tilde{\gamma}_{\tilde{x}}(0) = \tilde{x}$. This assignment is precisely the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, and as such, for any $[\gamma] \in \pi_1(X, x_0)$, the order of $F_{[\gamma]}$ must divide $|\pi_1(X, x_0)| = |\mathbb{Z}_5| = 5$.

- Suppose the cover p is nontrivial. Then there's some $[\gamma] \in \pi_1(X, x_0)$ such that the order of $F_{[\gamma]}$ is not 1. Then by the above, this order must be 5. As such, $F_{[\gamma]}$ is a permutation of 5 distinct elements of $p^{-1}(x_0)$ belonging to a single connected component of $\tilde{X}$. But $\left|p^{-1}(x_0)\right| = 4$, so this is impossible.

$\square$

Problem 4.

Background. A *contact manifold* is a pair (M^{2m+1}, ξ) consisting of an odd-dimensional manifold M together with a "maximally nonintegrable" field of hyperplanes $\{\xi_x \subset \mathsf{T}_x M\}_{x\in M}$, that is, a rank-$2m$ distribution ξ on M which is the kernel of some 1-form $\alpha \in \Omega^1(M)$, called a *contact form*, satisfying $\alpha \wedge (\mathsf{d}\alpha)^{\wedge m} \neq 0$ at each point of M. In this problem we show that $(\mathbb{R}^3, \mathscr{D})$ is a contact manifold.

No. It's enough to show that $\mathscr{D}$ is nonintegrable at $0 \in \mathbb{R}^3$. Let $\alpha := 2\mathsf{d}x - e^y\mathsf{d}z$, so that $\mathscr{D} = \mathsf{ker}(\alpha)$. It's a basic fact from contact geometry that $\mathscr{D}$ is nowhere integrable if $\alpha \wedge (\mathsf{d}\alpha) \neq 0$ at every point of $\mathbb{R}^3$. Indeed,

$$\alpha \wedge (\mathsf{d}\alpha) = (2\mathsf{d}x - e^y\mathsf{d}z) \wedge (-e^y\mathsf{d}y \wedge \mathsf{d}z) = -2e^y\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$$

is nonzero at every point of $\mathbb{R}^3$, and in particular at 0. □

Problem 5.

Suppose M is a submanifold of $\mathbb{R}^4$, and observe that $M = f^{-1}(0)$ where $f : \mathbb{R}^4 \to \mathbb{R}$ is the map given by $f(x_1, x_2, x_3, x_4) := x_1^2 + x_2^2 - x_3^2 - x_4^2$. Consider the tangent spaces of M at two of its points, 0 and $(1, 0, 1, 0)$,

$$\begin{aligned}
&\mathsf{T}_0 M = \mathsf{ker}(\mathsf{d}f_0) = \mathsf{ker}\begin{pmatrix}2x_1 & 2x_2 & 2x_3 & 2x_4\end{pmatrix}\big|_0 = \mathsf{ker}(0) = \mathsf{T}_0\mathbb{R}^4,\\
&\mathsf{T}_{(1,0,1,0)} M = \mathsf{ker}(\mathsf{d}f_{(1,0,1,0)}) = \mathsf{ker}\begin{pmatrix}2x_1 & 2x_2 & 2x_3 & 2x_4\end{pmatrix}\big|_{(1,0,1,0)} = \mathsf{ker}\begin{pmatrix}2 & 0 & -2 & 0\end{pmatrix}\\
&\quad = \{(v_1, v_2, v_3, v_4) \in \mathsf{T}_{(1,0,1,0)}\mathbb{R}^4 \mid 2v_1 - 2v_3 = 0\}.
\end{aligned}$$

Then $\mathsf{dim}_{\mathbb{R}}(\mathsf{T}_0 M) = 4$ but $\mathsf{dim}_{\mathbb{R}}(\mathsf{T}_{(1,0,1,0)} M) = 3$, which is impossible. □

Problem 6.

Let U and V be the cylinders along the z- and y-axes, respectively. Then $U \cap V \cong \mathsf{S}^1 \coprod \mathsf{S}^1$, so we have

$$\mathsf{H}_j(U) \cong \mathsf{H}_j(V) \cong \begin{cases}\mathbb{Z} & j = 0, 1,\\ 0 & \text{else,}\end{cases}, \quad \mathsf{H}_j(U \cap V) \cong \begin{cases}\mathbb{Z}^{\oplus 2} & j = 0, 1,\\ 0 & \text{else.}\end{cases}$$

By path connectedness, we already have $\mathsf{H}_0(X) \cong \mathbb{Z}$. Then by Mayer-Vietoris, the sequence

$$0 \longrightarrow \mathsf{H}_2(X) \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 2} \xrightarrow{(i_1, j_1)} \mathbb{Z}^{\oplus 2} \xrightarrow{k_1 - \ell_1} \mathsf{H}_1(X) \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2}$$

is exact.

- By exactness, $\mathsf{ker}(\partial_2) \cong 0$. Now consider the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$. The two loops x, y generating $\mathsf{H}_1(U \cap V)$ are mapped under i into contractible portions of the wall of the cylinder U, and so $i_1(x) = i_1(y) = 0$. On the other hand, j sends these two loops to the (same) single loop which generates $\mathsf{H}_1(V)$, and so $j_1(x) = j_1(y) = 1$. Thus $\mathsf{im}(\partial_2) \cong \mathsf{ker}(i_1, j_1) \cong \mathbb{Z}$, and so $\mathsf{H}_2(X) \cong \mathbb{Z}$.
- By the above, $\mathsf{ker}(k_1 - \ell_1) \cong \mathsf{im}(i_1, j_1) \cong \mathbb{Z}$, and so $\mathsf{ker}(\partial_1) \cong \mathsf{im}(k_1 - \ell_1) \cong \mathbb{Z}$. Next observe that the two connected components which together generate $\mathsf{H}_0(U \cap V)$ are mapped by i to the (same) single connected component of U which generates $\mathsf{H}_0(U)$. A similar statement holds for j, whereby $\mathsf{im}(i_0, j_0) \cong \mathbb{Z}$ and $\mathsf{im}(\partial_1) \cong \mathsf{ker}(i_0, j_0) \cong \mathbb{Z}$. Thus $\mathsf{H}_1(X) \cong \mathbb{Z}^{\oplus 2}$.

Hence $\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 7.

This is very similar to problem 5 of 2010, Fall, and problem 5 of 2008, Spring.