Fall 2015 $\boxed{\#1}$

(a) Let $f, g : X \to Y$. We say $H : X \times I \to Y$ is a homotopy between f and g if H continuous, $H(x,0) = f(x)$, $H(x,1) = g(x)$. $(f \simeq g)$

We say $f : X \to Y$ is a homotopy equivalence between X and Y if $\exists\, g : Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. $(X \simeq Y)$

(b) $\mathbb{R} \simeq \{0\} \subset \mathbb{R}$ by def. retract but no bijective map between spaces, so $\mathbb{R} \not\cong \{0\}$.

(c) S^2 and S^3 have trivial fundamental group but they are not homeomorphic because $H_2(S^2) = \mathbb{Z}$, $H_2(S^3) = 0$.

(d) $S^1 \vee S^1$ and $T^2 = S^1 \times S^1$.

$H_1(S^1 \vee S^1) = H_1(S^1) \oplus H_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} = H_1(T^2)$

but $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, while $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.

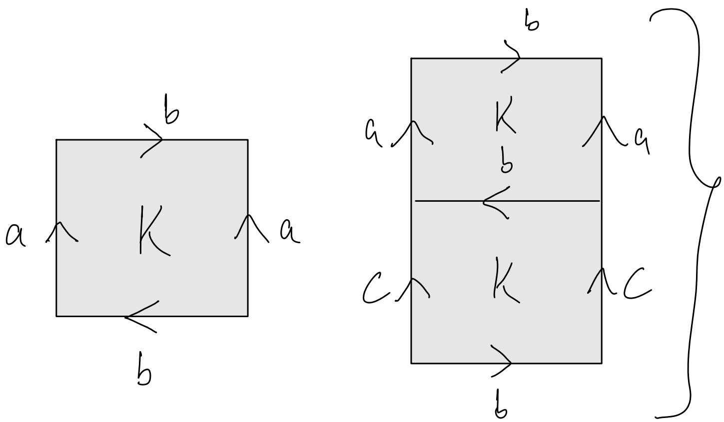
Fall 2015 $\boxed{\#2}$

$T = S^1 \times S^1$, $K =$ Klein Bottle.

(a) describe 2-sheeted covering $p: T \longrightarrow K$.

(b) $x_0 \in T$, $y_0 = p(x_0) \in K$. Give generators for $\pi_1(T; x_0)$, $\pi_1(K; y_0)$. For each generator of $\pi_1(T; x_0)$ express its image under $p_*: \pi_1(T; x_0) \to \pi_1(K; y_0)$.

(a)



T

map $a \mapsto a$
$b \mapsto b$
$c \mapsto a$

(b) We see $\pi_1(T; x_0) = \mathbb{Z} \times \mathbb{Z}\ \langle ca, b \rangle$

$\pi_1(K; p(x_0)) = \langle a, b \mid aba^{-1}b = 1 \rangle$

and $p_*(ca) = a^2$, $p_*(b) = b$

Fall 2015 $\boxed{\#3}$

$\Sigma_g, \Sigma_{g'}$ closed orientable surfaces of genus $g, g' > 0$.

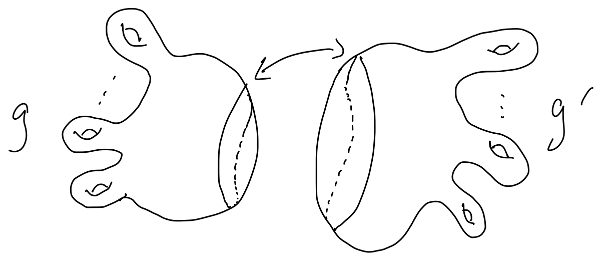
$f: B^2 \to \Sigma_g$ embedding. Simple closed curve $\gamma = f(S^1) \subset \Sigma_g$.

$\gamma' = f'(S^1) \subset \Sigma_{g'}$ with $f': B^2 \to \Sigma_{g'}$.

X = top. space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'. I.e. $X = \Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ $\forall x \in S^1$.

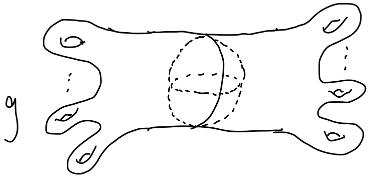
(a) compute $\pi_1(X)$.

f embedding $\Rightarrow$ $\exists$ homotopy from γ to the constant map at $\gamma(0)$. Same for γ'.

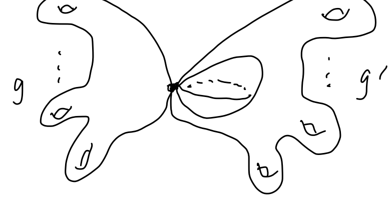


$\Sigma_g \vee \Sigma_{g'} \vee S^2$

$\parallel$



$\simeq$



Hence $\pi_1(X) \approx \pi_1(\Sigma_g) * \pi_1(\Sigma_{g'}) * \pi_1(S^2)$

$$= \boxed{(\mathbb{Z} \times \mathbb{Z})^{*g+g'}}$$

(b) $\tilde{H}_n(X) \approx \tilde{H}_n(\Sigma_g) \oplus \tilde{H}_n(\Sigma_{g'}) \oplus \tilde{H}_n(S^2)$

$$= \begin{cases} 0 & n=0 \\ \mathbb{Z}^{\oplus 2(g+g')} & n=1 \\ \mathbb{Z}^{\oplus 3} & n=2 \\ 0 & n>2 \end{cases}$$

(c) No because $H_2(X) \neq H_2(\Sigma_g \times \Sigma_{g'})$.

2015 Fall $\boxed{\#4}$

M n-dim'l mfd. $\omega \in \Omega^{n-1}(M)$. $\int_N \omega = 0$ $\forall$ $(n-1)$-dim'l oriented closed submfd N of M.

Claim: $d\omega = 0$.

Let $x \in M$. There is a chart centered at x, call it (U, ϕ). So $\phi : U \longrightarrow \mathbb{R}^n$ with $\phi(x) = 0$.

$\exists$ small $(n-1)$-dim'l sphere $S^{n-1} \subset \phi(U)$ centered at 0.

Let this be the boundry of a small B^n. Then $\phi^{-1}(B^n)$ contains x in M with boundary $\phi^{-1}(S^{n-1})$.

This $\phi^{-1}(S^{n-1})$ is a $(n-1)$-dim'l oriented closed submfd by regular level set theorem wrt the radius function. Then $\int_{\phi^{-1}(S^{n-1})} \omega = \int_{\phi^{-1}(B^n)} d\omega = 0$

by Stokes. Since x arbitrary, $\exists$ a collection of such $\phi^{-1}(B^n)$ covering M. Hence $d\omega = 0$ on M,

For if not, then $\exists$ $\phi^{-1}(B^n)$ on which $\int_{\phi^{-1}(B^n)} d\omega \neq 0$.

2015 Fall $\boxed{\#5}$

V.F. $v = \partial_x + xz\partial_z$, $w = \partial_y + yz\partial_z$ in $\mathbb{R}^3$.

$p \in \mathbb{R}^3$. $\exists$? local coordinate system in a nbd of p in with v and w?

I.e. is there a diffeomorphism $\phi : U \to V$ from a nbd U of p to an open $V \subset \mathbb{R}^3$ that sends v to ∂_x and w to ∂_y?

We need ϕ s.t. $d\phi_p(\partial_x + xz\partial_z) = \partial_x$

and $d\phi_p(\partial_y + yz\partial_z) = \partial_y$.

i.e.

$$\begin{pmatrix} \phi^1_x & \phi^1_y & \phi^1_z \\ \phi^2_x & \phi^2_y & \phi^2_z \\ \phi^3_x & \phi^3_y & \phi^3_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ xz \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \phi^1_x & \phi^1_y & \phi^1_z \\ \phi^2_x & \phi^2_y & \phi^2_z \\ \phi^3_x & \phi^3_y & \phi^3_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ yz \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So $\phi^1_x + xz\phi^1_z = 1$, $\phi^2_x + yz\phi^2_z = 1$

$$\phi_x^2 + xz\phi_z^2 = 0 \Rightarrow -xz\phi_z^2 + yz\phi_z^2 = 1$$

$$\Rightarrow \phi_z^2 = \frac{1}{z(y-x)}$$

$$\Rightarrow \phi^2 = \frac{1}{y-x}\ln(z) + C$$

$$\Rightarrow \phi_x^2 = -xz\phi_z^2 = -xz\frac{1}{z(y-x)} = -\frac{x}{y-x} = \frac{x}{x-y}$$

But $\phi_x^2 = -\frac{1}{(y-x)^2}\ln(z) \Rightarrow \Leftarrow$.

$$[V,W] = VW - WV = (\partial_x + xz\partial_z)(\partial_y + yz\partial_z) - WV$$

$$= \partial_x(\partial_y + yz\partial_z) + xz\partial_z(\partial_y + yz\partial_z) - WV$$

$$= \partial_{xy} + yz\partial_{xz} + xz\partial_{yz} + xyz\partial_z + xyz^2\partial_{z^2} - WV$$

$$= \partial_{xy} + yz\partial_{xz} + xz\partial_{yz} + xyz\partial_z + xyz^2\partial_{z^2}$$
$$- (\partial_y + yz\partial_z)(\partial_x + xz\partial_z)$$

$$= \cancel{\partial_{xy}} + \cancel{yz\partial_{xz}} + \cancel{xz\partial_{yz}} + \cancel{xyz\partial_z} + \cancel{xyz^2\partial_{z^2}}$$
$$\cancel{-\partial_{xy}} - \cancel{xz\partial_{yz}} - \cancel{yz\partial_{xz}} - \cancel{xyz\partial_z} - \cancel{xyz^2\partial_{z^2}}$$

$$= 0$$

2015 Fall $\boxed{\#6}$

$f: \mathbb{C} \to \mathbb{C}$ polynomial of one complex variable.

One point compactification $\mathbb{C} \cup \{\infty\} \cong S^2$.

(a) Prove f extends to a continuous $\bar{f}: S^2 \to S^2$.

(b) Prove $\deg(\bar{f}) = \deg(f)$.

(a) extend to $\bar{f}$ by sending ∞ to ∞. We have to check that $\bar{f}$ is continuous.

$$\begin{tikzcd}
\mathbb{C} \cup \{\infty\} \arrow[r, "{z \mapsto f(z)}", "{\infty \mapsto \infty}"'] \arrow[d, "\cong"] & \mathbb{C} \cup \{\infty\} \arrow[d, "\cong"] \\
S^2 \arrow[r, "\bar{f}"] & S^2
\end{tikzcd}$$

if $z \to \infty$ does $f(z) \to \infty$?

$f(z) = W_0 z^0 + w_1 z^1 + \cdots + w_d z^d$ for $d = \deg(f)$.

$$f(re^{i\theta}) = W_0 r + w_1 r e^{i\theta} + \cdots + w_d r^d e^{i\theta d}$$

$$= r^d \left(W_0 \frac{1}{r^{d-1}} + w_1 \frac{1}{r^{d-2}} e^{i\theta} + \cdots + w_d e^{i\theta d} \right)$$

$$\lim_{r \to \infty} f(re^{i\theta}) = \lim_{r \to \infty} r^d w_d e^{id\theta} = \infty,$$ So $\bar{f}$ Continuous.

(b) $\bar{f}_* : H_2(S^2) \to H_2(S^2)$ is multiplication by $n \in \mathbb{Z}$.
$\quad\;\; \mathbb{Z} \longrightarrow \mathbb{Z}$ $\quad \Rightarrow \deg(\bar{f}) = n$.

Geometrically, $\int_{S^2} \bar{f}^* \omega = n \int_{S^2} \omega$ for $\omega \in \Omega^2(S^2)$.

$$(\bar{f}^* \omega)_y(V_1, V_2) = \left(d\bar{f}_y(\omega)(V_1),\ d\bar{f}_y(\omega)(V_2) \right)$$

$$d\bar{f}_y = W_1 + 2w_2 y^1 + \cdots + d w_d \quad {}^{d-1}$$