Geometry and Topology Graduate Exam
Fall 2015

Problem 1. (15 points)

(a) Define the two notions "homotopy between two maps" and "homotopy equivalences between two spaces".

(b) Give an example of two topological spaces X and Y that are homotopy equivalent but are not homeomorphic.

(c) Give an example of path-connected topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.

(d) Give an example of path-connected topological spaces X and Y that have isomorphic first homology groups $H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but whose fundamental groups are not isomorphic.

Problem 2. (15 points) Let T be the 2–dimensional torus, and let K be the Klein bottle.

(a) Describe a twofold covering map $p\colon T \to K$. ("Twofold" means that the preimage of each point of K consists of two points of T.)

(b) Pick base points $x_0 \in T$ and $y_0 \in K$ such that $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T;x_0)$ and $\pi_1(K;y_0)$ and, for each generator of $\pi_1(T;x_0)$, express its image under the induced homomorphism $p_*\colon \pi_1(T;x_0) \to \pi_1(K;y_0)$ in terms of the generators of $\pi_1(K;y_0)$.

Problem 3. (25 points) Let Σ_g and $\Sigma_{g'}$ be closed orientable surfaces of genus g and $g' > 0$, respectively. Let $f\colon B^2 \to \Sigma_g$ be an embedding of the 2-dimensional disk B^2, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f'\colon B^2 \to \Sigma_{g'}$. Finally, let X be the topological space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'; namely, X is obtained from the disjoint union $\Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for every $x \in S^1$.

(a) Compute the fundamental group of X.

(b) Compute all homology groups of X.

(c) Is X homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

Problem 4. (15 points) Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$–dimensional oriented closed submanifold N of M. Show that $d\omega = 0$. (Possible hint: look at small spheres.)

Problem 5. (15 points) Consider the vector fields $\mathbf{v} = \partial_x + xz\partial_z$ and $\mathbf{w} = \partial_y + yz\partial_z$ in $\mathbb{R}^3$. If P is a point of $\mathbb{R}^3$, does there exist a local coordinate system in a neighborhood of P in which $\mathbf{v}$ and $\mathbf{w}$? Namely, is there a diffeomorphism $\phi\colon U \to V$ from a neighborhood U of P to an open subset $V \subset \mathbb{R}^3$ that sends $\mathbf{v}$ to ∂_x and $\mathbf{w}$ to ∂_y?

Problem 6. (15 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of one complex variable. Recall that the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is homeomorphic to the sphere S^2.

(a) Show that f extends to a continuous map $\bar{f}\colon S^2 \to S^2$.

(b) Show that the degree of $\bar{f}$ (in the sense of topology or geometry) is equal to the degree of the polynomial f (in the algebraic sense).

Geometry Fall 2015

① (a) Define "homotopy between two maps" and "homotopy equivalences between two spaces".

---

solution to (a)

Two maps $f, g : X \to Y$ between top spaces X, Y are homotopic if there is a map $F : X \times [0,1] \to Y$ s.t. $F(x,0) = f(x)\ \forall x \in X$, and $F(x,1) = g(x)\ \forall x \in X$; F is called a homotopy from f to g; we write $f \simeq g$.

Two top spaces X and Y are homotopy equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ s.t. $fg \simeq id_Y$ and $gf \simeq id_X$; we write $X \simeq Y$. □

(b) Give X, Y homotopy equivalent but not homeomorphic.

---

solution to (b)

$$\mathbb{R}^n \simeq \mathbb{R}^0 \quad (n \geq 0)$$

but $$\mathbb{R}^n \not\cong \mathbb{R}^0 \quad (n > 0).$$ □

(c) Give X, Y path connected w/ isomorphic fundamental groups that are not homotopy equivalent.

---

solution to (c)

$$X = \mathbb{R}^0, \quad Y = S^n \quad (n \geq 2).$$

□

(d) Give X, Y path connected s.t.

$H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but

$\pi_1(X) \not\cong \pi_1(Y)$.

---

solution to (d)

$\pi_1(\infty) \cong F_2$.

$\pi_1(\text{figure-8}) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_1(\infty;\mathbb{Z}) \cong H_1(\text{figure-8};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

□

---

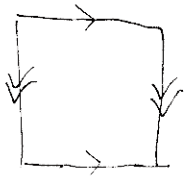
(2) T is the 2-dim torus.

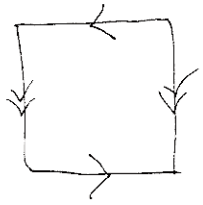
K is the Klein bottle.

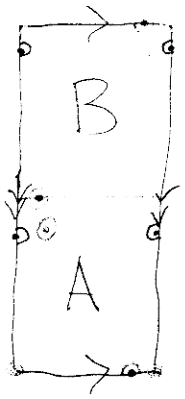
(a) Describe a two-fold covering map $p: T \to K$.

---

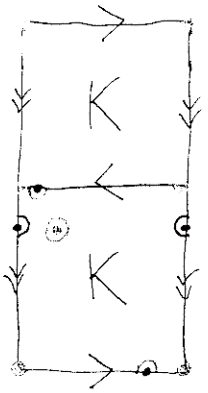
solution to (a)

T : 

K : 



p

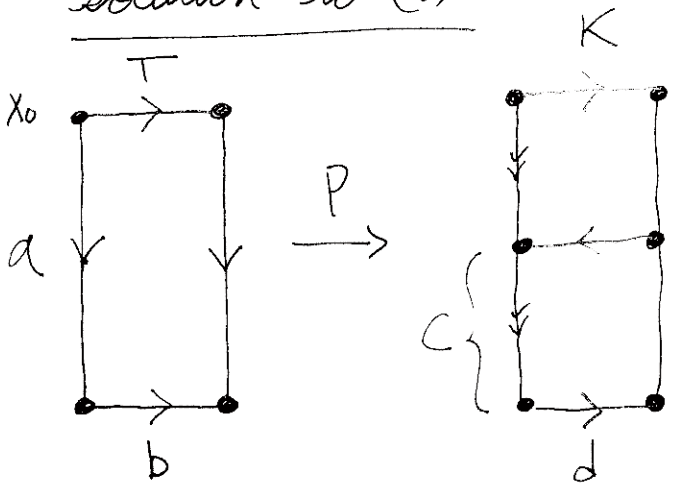


$B \to K$ is well-defined. ✓ It is continuous because it is the composition of a continuous map followed by a quotient map. ✓ Similarly, $A \to K$ is w.d. & continuous. ✓
By the pasting lemma this induces a continuous map $T \to K$ that is two-fold. ✓ By inspection, p is a covering map. □

(b) Pick base points $x_0 \in T$ and $y_0 \in K$ s.t. $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T; x_0)$ and $\pi_1(K; y_0)$. For each generator of $\pi_1(T; x_0)$ express its image under the induced homomorphism $p_* : \pi_1(T; x_0) \to \pi_1(K; y_0)$ in terms of the generators of $\pi_1(K; y_0)$.

---

solution to (b)



$p_*([a]) = 2[c]$

$p_*([b]) = [d]$

$\pi_1(T; x_0) = \langle a, b \mid aba\bar{b}\bar{b}^{-1} = 1 \rangle$ $\pi_1(K; y_0) = \langle c, d \mid dcdc^{-1} = 1 \rangle$

□

(3) Σ_g, $\Sigma_{g'}$ closed orientable surfaces of genus $g, g' > 0$, respectively.

Let $f: B^2 \longrightarrow \Sigma_g$ be an embedding of the 2-dim disk B^2, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$.

Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f': B^2 \to \Sigma_{g'}$.

Finally, let X be the topological space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'; namely, X is obtained from the disjoint union $\Sigma_g \amalg \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for all $x \in S^1$.

(a) Compute the fundamental group of X.

(b) Compute all homology groups of X.

(c) Is X homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

solution to (a)

$$\pi_1(\Sigma_g; x_0) = \left\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \;\middle|\; a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1} = \dots = a_g b_g a_g^{-1} b_g^{-1} = 1 \right\rangle$$

$$\pi_1(\Sigma_{g'}; x_0') = \left\langle c_1, d_1, c_2, d_2, \dots, c_{g'}, d_{g'} \;\middle|\; c_1 d_1 c_1^{-1} d_1^{-1} = c_2 d_2 c_2^{-1} d_2^{-1} = \dots = c_{g'} d_{g'} c_{g'}^{-1} d_{g'}^{-1} = 1 \right\rangle$$

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^{2g}, & k = 1 \\ 0, & k > 2 \end{cases} \qquad H_k(\Sigma_{g'}) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^{2g'}, & k = 1 \\ 0, & k > 2 \end{cases}$$

we may assume $x_0 \in \gamma$, $x_0' \in \gamma'$

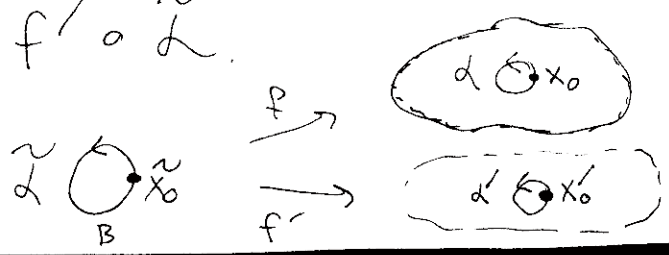
and $p(x_0) = p(x_0')$ where p is the quotient map

$\Sigma_g \amalg \Sigma_{g'} \longrightarrow X$. Let $\tilde{\alpha}$ be a generator

for S^1 s.t. $f(\tilde{\alpha}(0)) = f(\tilde{x}_0) = x_0$ and

$f'(\tilde{\alpha}(0)) = f'(\tilde{x}_0') = x_0'$; let $\alpha := f \circ \tilde{\alpha}$ and

$\alpha' := f' \circ \tilde{\alpha}$.



We claim $p \circ \alpha = p \circ \alpha'$ as a path in X. Indeed,

$$p \circ \alpha(t) = p \circ \alpha'(t) \in X \iff$$

$$\alpha'(t) = f' \circ f^{-1} \circ \alpha(t)$$

$$\| \qquad\qquad \|$$

$$f' \circ \tilde{\alpha}(t) \qquad f' \circ f^{-1} \circ f \circ \tilde{\alpha}(t). \quad \checkmark$$

Let $X_1 = p(\Sigma_g) \subset X$, $X_2 = p(\Sigma_{g'}) \subset X$.

Then $[\xi := [p \circ \alpha] = [p \circ \alpha']$ is a generator for $\pi_1(X; z_0)$ where $z_0 := p(x_0) = p(x_0')$.

By the SvK closed version (It is obvious X is Hausdorff, so p is a closed map, as $\Sigma_1 \amalg \Sigma_2$ is compact, so X_1 & X_2 are closed; $X_1 \cap X_2 = \operatorname{im} \xi \subset X$ is p.c. & clearly a nbhd def retract in X_1 & X_2.)

$$\pi_1(X; z_0) \cong \pi_1(X_1; z_0) *_{\pi_1(X_1 \cap X_2; z_0)} \pi_1(X_2; z_0)$$

where the amalgamation occurs along

$(i_j)_* : \pi_1(X_1 \cap X_2; z_0) \to \pi_1(X_j; z_0) \quad (j=1,2)$.

Therefore,

$$\pi_1(X; z_0) \cong \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, d_1, c_2, d_2, \ldots, c_{g'}, d_{g'} \mid$$

$$a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1} = \cdots = a_g b_g a_g^{-1} b_g^{-1} = c_1 d_1 c_1^{-1} d_1^{-1} = c_2 d_2 c_2^{-1} d_2^{-1} = \cdots = c_{g'} d_{g'} c_{g'}^{-1} d_{g'}^{-1}$$

$$= (i_1)_*([\xi])(i_2)_*^{-1}([\xi]) = 1 \rangle$$

where the last term goes to 1 since $(i_1)_*([\xi]) = [\alpha] = 1$ as α is the boundary of an embedded disk, and similarly for $(i_2)_*([\xi])$. $\square$

solution to (b)

Use also $H_k(S^1) = \begin{cases} \mathbb{Z}, & k=0,1 \\ 0, & k>1 \end{cases}$.

Since the hypotheses of Ex 4 were satisfied, so are those for MV (recall for MV you don't need $X_1 \cap X_2$ p.c., in fact). So we get LES

$$\xrightarrow{\delta_{k+1}} H_k(X_1 \cap X_2) \xrightarrow{(-H_k(i_1),\, H_k(i_2))} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(j_1)+H_k(j_2)} H_k(X_1 \cup X_2) \xrightarrow{\delta_k} H_{k-1}(X_1 \cap X_2) \to \cdots$$

$k>2$:
$$\cdots \xrightarrow{\delta_{k+1}} 0 \to 0 \oplus 0 \to H_k(X) \to 0 \xrightarrow{\delta_k} \cdots$$

$$\therefore \boxed{H_k(X) = 0}$$

$k=2$:
$$\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} H_2(X) \xrightarrow{g \circ \delta_2} \mathbb{Z} \xrightarrow{h \equiv 0} \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g-1} \to \cdots$$

$H_2(X) \cong \ker f \cong \mathbb{Z} \times \mathbb{Z}$

$$\therefore \boxed{H_2(X) \cong \mathbb{Z}^3}$$

* since image of 1-simplex on the boundary of Π a 0-cycle is contractible.

$k=1$:

$$\cdots \xrightarrow{\delta_2} \mathbb{Z} \xrightarrow{f=0} \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'} \xrightarrow{g} H_1(X) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z} \to \cdots$$

$$H_1(X) \Big/ \operatorname{img} \cong \frac{H_1(X)}{\ker \delta_1} \cong \operatorname{im}\delta_1 \cong \ker h \cong 0$$

$$\cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'}$$

$$\therefore \boxed{H_1(X) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'}}$$

$k=0$: $\boxed{H_0(X) \overset{p.c.}{\cong} \mathbb{Z}}$. □.

---

solution to (c)

no. $H_2(\Sigma_g \times \Sigma_{g'}) \cong \mathbb{Z} \times \mathbb{Z}$

$\not\cong \mathbb{Z}^3$, and homology is preserved under homotopy equivalence. □.

(4) M mfld dim n,

$\omega \in \Omega^{n-1}(M)$. Suppose $\int_N \omega = 0$ for all $(n-1)$-dimensional oriented closed submflds N of M. Show $d\omega = 0$.

solution Special case, $M = \mathbb{R}^n$

Lemma/ If $\alpha \in \Omega^n(\mathbb{R}^n)$ and $\int_{\overline{B(x,\varepsilon)}} \alpha = 0$ for all $B(x,\varepsilon) = \{y \in \mathbb{R}^n \mid \|x-y\| < \varepsilon\}$, then $\alpha = 0 \in \Omega^n(\mathbb{R}^n)$.

Pf(lemma)/ $\alpha = f(x_1,\dots,x_n)\, dx_1 \wedge \cdots \wedge dx_n$

for some smooth f. Since

$$\int_{\overline{B(X,\varepsilon)}} \alpha := \int_{\overline{B(X,\varepsilon)}} f(X_1,\dots,X_n)\,dX_1\cdots dX_n$$

$$\left(= \int_{B(X,\varepsilon)} f(X_1,\dots,X_n)\,dX_1\cdots dX_n \right)$$

↑ Lebesgue measure of $\overline{\partial B(X,\varepsilon)}$ is zero

$$= 0 \quad \text{for all } X, \varepsilon \;\Big)$$

continuity of f alone shows $f \equiv 0$ ✓

By Stokes' theorem,

$$\int_{\overline{B(X,\varepsilon)}} d\omega = \int_{\overline{\partial B(X,\varepsilon)}} \omega \overset{\text{hyp.}}{=} 0.$$

By the lemma, $dw \equiv 0$.

This completes the proof of the special case. ✓

General case, M arbitrary

$dw = 0$ is a local property.

By restricting w to a chart we are done by the special case. □

---

(5) Consider the vector fields

$$X = \frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$$

in $\mathbb{R}^3$. If p is a point of $\mathbb{R}^3$, is there a local coordinate system $(\tilde{x}, \tilde{y})$ around p s.t. $\frac{\partial}{\partial \tilde{x}} = X$ and $\frac{\partial}{\partial \tilde{y}} = Y$?

solution to (5)

A sufficient condition is

$$[X, Y] = 0.$$

I'm pretty sure, like 90%, that it is necessary.

↑

Yes. Since $[\partial_x, \partial_y] = 0$ always.

Recall $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.

Here, $Xg(p) := X_p(g) \in \mathbb{R}$.

$$[X, Y] = [\partial_x + xz\partial_z, \partial_y + yz\partial_z]$$

$$= [\partial_x, \partial_y] + [\partial_x, yz\partial_z] + [xz\partial_z, \partial_y] + [xz\partial_z, yz\partial_z]$$

$$= yz[\partial_x, \partial_z] + 1\cdot\partial_x(yz)\partial_z - yz\partial_z(1)\partial_x + xz[\partial_z, \partial_y] + xz\partial_z(1)\partial_y - \partial_y(xz)\partial_z$$

$$+ XZyz\left[\frac{\partial z}{\partial z}\right] + XZ\partial_z(yz)dz$$

$$- yz\partial_z(xz)dz$$

$$= XZy - yzX = 0.$$

$\therefore$ such a coordinate system exists. $\square$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a *polynomial* of one complex variable. Recall the one-point compactification $\mathbb{C} \cup \{\infty\}$ is homeomorphic to S^2.

(a) Show f extends to a continuous map $\bar{f} : S^2 \to S^2$.

(b) Show the degree of $\bar{f}$, in either the sense of topology or geometry, is equal to the degree of the polynomial f, in the sense of algebra.

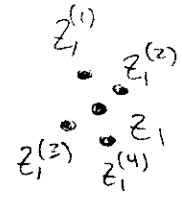
Solution to (a) & (b) (sketch)

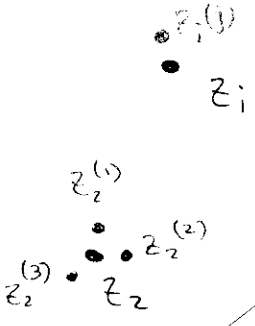
For (a), setting $\bar{f}(\infty) = \infty$

is clearly the desired extension.

For (b), there are regular values $w \in S^2$ very close to 0.

The preimages of this regular value form little groups $z_i^{(j)}$ around the various roots z_i, where each group has m_i elts if z_i has multiplicity m_i.





because the tangent map is orientation-preserving at every point

Since f is holomorphic, all the local degrees of the preimages are $+1$. We conclude the geometric degree is $\sum m_i = n$. □