2015, Fall

Problem 1.

(a)
- A *homotopy* between two continuous maps $f, g : X \to Y$ of topological spaces is a continuous map $h : X \times [0,1] \to Y$ with $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. In this case, f and g are said to be *homotopic*.
- Two topological spaces X, Y are said to be *homotopy equivalent* if there exists a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to id_X, and $f \circ g$ is homotopic to id_Y. In this case, f and g are called *homotopy equivalences* between X and Y.

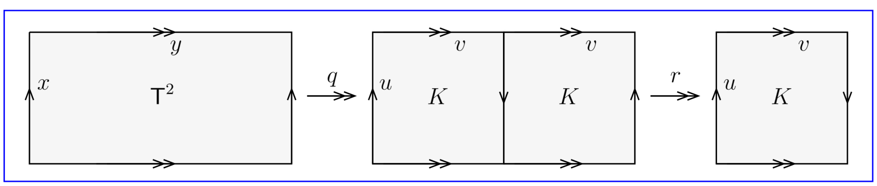
(b) The closed disc B^2 is homotopy equivalent to a point $*$ (since it's contractible), but B^2 and $*$ aren't homeomorphic since any map $\mathsf{B}^2 \to *$ is noninjective. ☐

(c) Both the sphere S^2 and the point $*$ have trivial fundamental group, but aren't homotopy equivalent since $*$ is contractible while S^2 isn't. ☐

(d) The torus T^2 and the wedge of two circles $\mathsf{S}^1 \vee \mathsf{S}^1$ both have first homology group isomorphic to $\mathbb{Z}^{\oplus 2}$, but the fundamental group of T^2 is the abelian group $\mathbb{Z}^{\oplus 2}$ while that of $\mathsf{S}^1 \vee \mathsf{S}^1$ is the nonabelian free group F_2. ☐

Problem 2.

(a) Let $p : \mathsf{T}^2 \twoheadrightarrow K$ be the composite of the quotient map q from T^2 to two Klein bottles K glued to one another as shown, followed by a projection r from this space onto a single copy of K.



This is the desired cover. ☐

(b) Let $x, y \in \pi_1(\mathsf{T}^2)$ and $u, v \in \pi_1(K)$ be loops in T^2 and K, respectively, corresponding to the edges above as labeled. We see that

$$\pi_1(\mathsf{T}^2) \cong \langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle, \quad \pi_1(K) \cong \langle u, v \mid uvuv^{-1} = 1 \rangle.$$

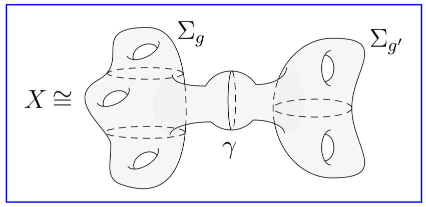
Moreover, by the diagram, $p_*(x) = u$ and $p_*(y) = r_*(q_*(y)) = r_*(2v) = 2r_*(v) = 2v$. ☐

Problem 3.

(a) Recall first that

$$\pi_1(\Sigma_g) \cong \left\langle x_i, y_i, 1 \le i \le g \,\middle|\, \prod_{i=1}^{g} [x_i, y_i] = 1 \right\rangle, \quad \pi_1(\Sigma_{g'}) \cong \left\langle u_j, v_j, 1 \le i \le g' \,\middle|\, \prod_{j=1}^{g'} [u_j, v_j] = 1 \right\rangle.$$

Within X, the copies of Σ_g and $\Sigma_{g'}$ intersect along the circular curve $\gamma = \gamma'$.



The inclusion $\gamma \hookrightarrow \Sigma_g$ induces the trivial homomorphism $\pi_1(\gamma) \to \pi_1(\Sigma_g)$ since, in Σ_g, the curve γ forms the boundary of an embedded (contractible) disc. By similar reasoning the inclusion $\gamma \hookrightarrow \Sigma_{g'}$ induces the trivial homomorphism $\pi_1(\gamma) \to \pi_1(\Sigma_{g'})$ as well, so by van Kampen

$$\pi_1(X) \cong \pi_1(\Sigma_g) * \pi_1(\Sigma_{g'}) \cong \left\langle x_i, y_i, u_j, v_j, 1 \le i \le g, 1 \le j \le g' \,\middle|\, \prod_{i=1}^{g}[x_i, y_i] = \prod_{j=1}^{g'}[u_j, v_j] = 1 \right\rangle.$$

□

(b) We already have by path connectedness that $\mathsf{H}_0(X) \cong \mathbb{Z}$, and by Hurewicz that

$$\mathsf{H}_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)] \cong \mathbb{Z}^{\oplus 2g} \oplus \mathbb{Z}^{\oplus 2g'} \cong \mathbb{Z}^{\oplus 2(g+g')}.$$

Letting U and V be, respectively, Σ_g and $\Sigma_{g'}$ extended slightly beyond γ within X, then $U \cong \Sigma_g, V \cong \Sigma_{g'}$, and $U \cap V \cong \mathsf{S}^1$. For any $j \ge 2$, Mayer-Vietoris immediately yields $\mathsf{H}_j(X) \cong 0$. We further have by Mayer-Vietoris the exact sequence

$$0 \longrightarrow \mathbb{Z}^{\oplus 2} \xrightarrow{f} \mathsf{H}_2(X) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^{\oplus 2(g+g')}.$$

By exactness, f is injective, whereby $\mathsf{ker}(\partial) \cong \mathsf{im}(f) \cong \mathbb{Z}^{\oplus 2}$. Moreover, as observed before, the map ι induced by the inclusions of γ into Σ_g and $\Sigma_{g'}$ is trivial, and so $\mathsf{im}(\partial) \cong \mathsf{ker}(\iota) \cong \mathbb{Z}$. Thus $\mathsf{H}_2(X) \cong \mathbb{Z}^{\oplus 3}$, and in summary

$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2(g+g')} & j = 1, \\ \mathbb{Z}^{\oplus 3} & j = 2, \\ 0 & \text{else}. \end{cases}$$

□

(c) **No**. We have $\mathsf{H}_2(\Sigma_g \times \Sigma_{g'}) \cong \mathsf{H}_2(\Sigma_g) \oplus \mathsf{H}_2(\Sigma_{g'}) \cong \mathbb{Z}^{\oplus 2}$, but we just showed that $\mathsf{H}_2(X) \cong \mathbb{Z}^{\oplus 3}$. Therefore $\Sigma_g \times \Sigma_{g'}$ and X can't be homotopy equivalent. □

Problem 4.

Assume $\mathsf{d}\omega \ne 0$. Then there's a point $x \in M$ at which $\mathsf{d}\omega_x \ne 0$. Let $U \subset M$ be a neighborhood of x homeomorphic to $\mathsf{B}^n \subset \mathbb{R}^n$ via some coordinate chart $\phi : U \to \mathsf{B}^n$ with local U-coordinates $y_1, \dots, y_n$. Then on U, we may write $\mathsf{d}\omega$ in the local form $\mathsf{d}\omega = f\mathsf{d}y_1 \wedge \ldots \wedge \mathsf{d}y_n$, for some

$f \in \mathsf{C}^\infty(U)$. Since $f(x)$ is nonzero, say w.l.o.g. $f(x) > 0$, then also w.l.o.g. U was chosen small enough so that $f > 0$ on all of U by continuity of f. Then

$$\int_U \mathsf{d}\omega = \int_{\mathsf{B}^n} (\phi^{-1})^*(\mathsf{d}\omega) = \int_{\mathsf{B}^n} \underbrace{(f \circ \phi^{-1})}_{>0} \mathsf{d}z_1 \wedge \ldots \wedge \mathsf{d}z_n > 0,$$

where $z_j =: \phi^* y_j$ is the B^n-coordinate corresponding to y_j, for each $1 \le j \le n$. But on the other hand, $\partial U \subset M$ is an oriented closed submanifold since it's homeomorphic to $\partial\mathsf{B}^n = \mathsf{S}^{n-1}$ via ϕ, so by Stokes and the problem assumption, $\int_U \mathsf{d}\omega = \int_{\partial U} \omega = 0$, a contradiction. □

Problem 5 (?)**.**

By Frobenius, it's enough to verify that $[v, w] = 0$. Now,

$$vw = \left(\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y} + xz\frac{\partial^2}{\partial y \partial z} + 0 + yz\frac{\partial^2}{\partial x \partial z} + xyz\frac{\partial}{\partial z} + xyz^2\frac{\partial^2}{\partial z^2},$$
$$wv = \left(\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y} + yz\frac{\partial^2}{\partial x \partial z} + 0 + xz\frac{\partial^2}{\partial y \partial z} + xyz\frac{\partial}{\partial z} + xyz^2\frac{\partial^2}{\partial z^2},$$

whereby $[v, w] = vw - wv = 0$. □

Problem 6.

Remark. In this problem, we use $\mathbb{C} \cup \{\infty\}$ and S^2 interchangeably by implicitly making use of the given homeomorphism. Note also that if $f : \mathbb{C} \to \mathbb{C}$ is a constant polynomial, then it trivially extends to S^2, and this extension has topological degree 0, which is the same as the algebraic degree of f. So we'll also assume w.l.o.g. that f is nonconstant.

(a) Define $\bar{f} : \mathsf{S}^2 \to \mathsf{S}^2$ by setting $\bar{f}\big|_{\mathbb{C}} := f$ and $\bar{f}(\infty) := \infty$. Clearly $\bar{f}$ is continuous on $\mathbb{C}$, so it remains to check continuity at ∞. Indeed, if $\{z_j\}_{j=1}^\infty \subset \mathsf{S}^2$ is a sequence converging to ∞, then

$$\lim_{j\to\infty} \bar{f}(z_j) = \lim_{j\to\infty} f(z_j) = \infty = \bar{f}(\infty),$$

where we have the third equality, by Liouville, since f is a nonconstant polynomial. □

(b) Say $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ for all $z \in \mathbb{C}$, where $a_0, \ldots, a_m \in \mathbb{C}$ and $a_m \neq 0$. Then the algebraic degree of f is $m \in \mathbb{N}$, and it's enough to show that $\bar{f}$ is homotopic to the map $g : \mathsf{S}^2 \to \mathsf{S}^2$ given by $g(z) := z^m$ for all $z \in \mathsf{S}^2$, since g has homological degree m. We begin with the map

$$h : \mathsf{S}^2 \times [0, 1] \to \mathsf{S}^2, \quad h(z, t) := t(a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}) + a_m z^m,$$

with $h_0(z) = a_m z^m$ for all $z \in \mathsf{S}^2$, and $h_1 = f$. Obviously h is continuous on $\mathbb{C} \times [0, 1]$, so it remains to check that it's also continuous at any point of the form $(\infty, t) \in \mathsf{S}^2 \times [0, 1]$.

Take any $(\infty, t) \in \mathsf{S}^2 \times [0, 1]$ and any $M > 0$. We need to check that there's some $K > 0$ large enough and $\delta > 0$ small enough so that whenever $(z, s) \in \mathsf{S}^2 \times [0, 1]$ has $|z| > K$ and $|s - t| < \delta$, then $|h(z, s)| > M$. But indeed, $|a_m z^m| > \left|a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}\right|$ whenever $|z| > K$ for some large $K > 0$, and so choosing this value of K together with $\delta := 1$ proves the desired continuity. Therefore h is a homotopy between f and h_0. Similarly, we can check that the map

$$k : \mathsf{S}^2 \times [0, 1], \quad k(z, t) := a_m^t z^m$$

is a homotopy between h_0 and g, and this completes the proof. □