Geometry/Topology Qualifying Exam - Fall 2014

$G(\tilde{X}) \cong \pi_1(X)$

#sheets $= |\pi_1(X)|$

1. Show that if (X, x) is a pointed topological space whose universal cover exists and is compact, then the fundamental group $\pi_1(X, x)$ is a finite group.

2. Recall that if (X, x) and (Y, y) are pointed topological spaces, then the wedge sum (or 1-point union) $X \vee Y$ is the space obtained from the disjoint union of X and Y by identifying x and y. Show that T^2 (the 2-torus $S^1 \times S^1$) and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but are not homeomorphic.

3. Suppose S^n is the standard unit sphere in Euclidean space and that $f : S^n \to S^n$ is a continuous map.

   construct homotopy

   degree $\Rightarrow \pm 1 \Rightarrow \varepsilon$

   i) Show that if f has no fixed points, then f is homotopic to the antipodal map.

   ii) Show that if $n = 2m$, then there exists a point $x \in S^{2m}$ such that either $f(x) = x$ or $f(x) = -x$.

Kunneth Formula.

4. If M is a smooth manifold of dimension d, using basic properties of de Rham cohomology, describe the de Rham cohomology groups $H^*_{dR}(S^1 \times M)$ in terms of the groups $H^*_{dR}(M)$ (along the way, please explain, quickly and briefly, how to compute $H^*_{dR}(S^1)$).

Gauss-Bonnet

5. Show that if $X \subset \mathbb{R}^3$ is a closed (i.e., compact and without boundary) submanifold that is homeomorphic to a sphere with $g > 1$ handles attached, then there is a non-empty open subset on which the Gaussian curvature K is negative.

stokes + cartan

6. Suppose M is a (non-empty) closed oriented manifold of dimension d. Show that if ω is a differential d-form, and X is a (smooth) vector field on X, then the differential form $\mathfrak{L}_X\omega$ necessarily vanishes at some point of M.

atlas

7. Let V be a 2-dimensional complex vector space, and write $\mathbb{CP}^1$ for the set of complex 1-dimensional subspaces of V. By explicit construction of an atlas, show that $\mathbb{CP}^1$ can be equipped with the structure of an oriented manifold.

Fall 2014

1) (X,x) is a pointed topological space whose universal cover exists & is compact then $\pi_1(X,x) < \infty$.

Pf Suppose $\tilde{X} \xrightarrow{p} X$ is the universal cover.

Then $\pi_1(\tilde{X}) = 0$ so $p_*(\pi_1(\tilde{x})) = 0$ is normal in $\pi_1(X)$.

So then $G(\tilde{X}) \simeq \pi_1(X)$. Now $G(\tilde{x})$ acts freely on $\tilde{X}$ & on the fiber of any $x \in X$, hence the number of sheets in the cover is precisely the number of elements in $G(\tilde{x})$ (since it acts transitively on the fiber of $x \in X$).

But $\tilde{X}$ is compact and the map $\tilde{X} \to X$ surjective so X is also compact, hence the cover is finitely sheeted $\Rightarrow \pi_1(X) < \infty$.

<u>Lemma</u>: $\tilde{X}$ compact & X compact $\Rightarrow$ cover is finitely sheeted.

Pf Suppose
$$\begin{array}{cc} \tilde{X} & \tilde{x}_i \\ \downarrow & \downarrow \cdots \\ X & x \end{array}$$
, then for any x, $\exists$ nbh such that $\mathcal{U}$ s/t $\tilde{\mathcal{U}}^x_\alpha = p^{-1}(\mathcal{U}) \simeq \mathcal{U}$. Now each α corresponds to a distinct sheet of the cover.

In particular, $\bigcup \tilde{\mathcal{U}}^x_\alpha$ form a cover for $\tilde{X}$, so $\exists$ a finite subcover, by compactness of $\tilde{X}$. $\Rightarrow$ there are finitely many $\tilde{\mathcal{U}}^x_\alpha$ so there are finitely many sheets.

(2) Show that $S^1 \times S^1$ & $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups but are not homeomorphic.

Pf. $H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\times\mathbb{Z} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & \text{else} \end{cases}$ (Recall $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$)

can show w/ Mayer Vietoris
let $A = S^1 \times S^1 \setminus \text{north pole} \simeq S^1$
$B = S^1 \times S^1 \setminus \text{south pole} \simeq S^1$

& $\tilde{H}_k(S^1 \vee S^1 \vee S^2) = \tilde{H}_k(S^1) \oplus \tilde{H}_k(S_1) \oplus \tilde{H}_k(S^2)$

(since all good pairs) $\Rightarrow$ $\tilde{H}_k(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\times\mathbb{Z} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & \text{else.} \end{cases}$

However $\pi_1(S^1 \times S^1) = \mathbb{Z}\times\mathbb{Z}$ but $\pi_1(S^1 \vee S^1 \vee S^2) = \mathbb{Z} * \mathbb{Z}$

so not homeomorphic since $\mathbb{Z}\times\mathbb{Z}$ is abelian but $\mathbb{Z} * \mathbb{Z}$ is not.

(3) Show that if $f: S^n \to S^n$ is a continuous map.

(i) if $f(x) \neq x \ \forall x \Rightarrow f \simeq \alpha : x \mapsto -x$

(ii) if $n = 2m$ then $\exists\, x \in S^{2m} \ni f(x) = x$ or $f(x) = -x$

Pf (i) $f(x) \neq x \Rightarrow h(t,x) = \dfrac{-xt + (1-t)f(x)}{|-xt + (1-t)f(x)|}$ is a homotopy

between f & α, since denominator is never zero.

(ii) if $f(x) \neq x$ & $f(x) \neq -x$ then f & $-f$ are both homotopic

to antipodal map $\Rightarrow$ $\deg f = (-1)^{2m+1}$ & $\deg(-f) = (-1)^{2m+1}$

$\Rightarrow \deg f = -1$ and $\deg f = 1$ $\Rightarrow \Leftarrow$.

So one of these must occur.

(4) M - dim d, smooth.

Describe $H^*_{dR}(S^1 \times M)$ in terms of $H^*_{dR}(M)$, & how to obtain $H^*_{dR}(S^1)$.

Pf: Recall the Künneth formula:

$$H^*_{dR}(S^1 \times M) \cong H^*_{dR}(S^1) \otimes_{\mathbb{R}} H^*_{dR}(M)$$

but $H^*_{dR}(S^1 \times M)$ is graded so:

$$H^n_{dR}(S^1 \times M) = \bigoplus_{i+j=n} \left(H^i_{dR}(S^1) \otimes_{\mathbb{R}} H^j_{dR}(M) \right)$$

However $H^i_{dR}(S^1) = \begin{cases} \mathbb{R} & i = 0, 1 \\ 0 & \text{else} \end{cases}$

$$\text{so} \quad H^n_{dR}(S^1 \times M) \cong \left(\mathbb{R} \otimes_{\mathbb{R}} H^n_{dR}(M) \right) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} H^{n-1}_{dR}(M) \right)$$

$$H^n_{dR}(S^1 \times M) \cong H^n_{dR}(M) \oplus H^{n-1}_{dR}(M)$$

Now for S^1 note that $H^0_{dR}(S^1) = \ker d : \Omega^0 \to \Omega^1$

i.e. $\{f ; df = 0\} \Rightarrow \{f = \text{constant}\} = \mathbb{R}$.

and if $w = d\theta$ - an orientation on S^1, then $\int_{S^1} d\theta = 2\pi$

so then for any other $\eta \in \Omega^1(S^1)$, we have that if $c = \frac{1}{2\pi} \int_{S^1} \eta$

then $\int_{S^1} (\eta - c \cdot w) = 0 \Rightarrow [\eta - cw] \equiv 0 \Rightarrow [\eta] = c \cdot [w]$.

That is $[w]$ generates $H^1_{dR}(S^1)$. So $H^1_{dR}(S^1) = \mathbb{R}$.

(5) $X \subseteq \mathbb{R}^3$ closed (compact w/o boundary) submanifold homeomorphic to sphere w/ $g > 1$ handles attached, then $\exists$ nonempty open subset on which gaussian curvature is neg.

Pf $X \simeq$ surface of genus g.

so by Gauss-Bonnet theorm:

$$\int_X K(g) = 2\pi\, \chi(X) = 2\pi(2-2g) < 0. \text{ since } g < 1$$

so then the set on which $K(g) < 0$ has nonempty interior

in particular, $\exists\, p$ & $B_{\epsilon,p} \ni p \ni K(x) < 0 \quad \forall x \in B_{\epsilon,p}$.

(6) M- nonempty closed oriented manifold of dimension d

Show if ω is a d-form & X-smooth v.f. on X then $\mathcal{L}_X\omega$ necessarily vanishes everywhere.

Pf Recall cartan's formula & stoke's theorm:

$$\int_M \mathcal{L}_X \omega = \int_M i_X(d\omega) + d\, i_X(\omega) = \int_M d\, i_X(\omega)$$

$d\omega = 0$ since M is d-dim so $\Omega^{n+1}(M) = 0$ & ω is a d-fm

$$\text{so by stokes} = \int_{\partial M} i_X(\omega) = \int_{\emptyset} i_X\omega = 0.$$

Since $M \neq \emptyset$ then $\mathcal{L}_X\omega = 0$ somewhere by continuity of the Lie derivative.

) $\mathbb{CP}^1 = \{1$ Dimensional subspaces of $\mathbb{C}^2\}$. = lines in $\mathbb{C}^2$

each line has equation $az_1 + bz_2 = 0$; $a, b \in \mathbb{C}$, not a or $b \neq 0$.

So each line is representable by $[a,b]$ where $[a,b] \sim [\lambda a, \lambda \beta]$ $\forall\ \lambda \in \mathbb{C} \setminus \{0\}$.

So consider $\mathcal{U}_1 = \{[a,b] \ni a \neq 0\}$, $\mathcal{U}_2 = \{[a,b] ; b \neq 0\}$.

Then, $\mathbb{C}^2_{\setminus\{0\}} \xrightarrow{\Pi} \mathbb{CP}^1$ then $\Pi^{-1}(\mathcal{U}_1) = \mathbb{C}^2 - \mathbb{C} \simeq \mathbb{C}$ is open

$(a,b) \longmapsto [a,b]$ $\quad \Pi^{-1}(\mathcal{U}_2) \cong \mathbb{C}$ also open.

Since projections are open maps $\Rightarrow \mathcal{U}_1, \mathcal{U}_2$ are open & cover $\mathbb{CP}^1$.

Now, consider $\varphi_1 : \mathcal{U}_1 \xrightarrow{b \neq 0} \mathbb{C}$ then φ_1 is invertible since

$[a,b] \longrightarrow \frac{a}{b}$ for any $c \in \mathbb{C} \xrightarrow{\varphi_1^{-1}} [c,1]$

Likewise $\varphi_2 : \mathcal{U}_2 \xrightarrow{a \neq 0} \mathbb{C}$ likewise invertible. $\varphi_2^{-1}(d) = [1, d]$

$[a,b] \longrightarrow \frac{b}{a}$

So then $\varphi_2^{-1} \circ \varphi_1 : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow \mathcal{U}_2$, so $a, b \neq 0$

$\varphi_2^{-1}\varphi_1([a,b]) = \varphi_2^{-1}\left(\frac{a}{b}\right) = [1, \frac{a}{b}] = [b, a] \in \mathcal{U}_2$ ✓

$\varphi_1^{-1}\varphi_2([a,b]) = \varphi_1^{-1}\left(\frac{b}{a}\right) = [b/a, 1] = [b, a] \in \mathcal{U}_1$ ✓

smooth anywhere on $\mathcal{U}_1 \cap \mathcal{U}_2$.

Now by the quotient topology $\Rightarrow \mathbb{CP}^1$ is Hausdorff & 2nd countable, as well as orientable.