Fall 2014 $\boxed{\#1}$

Universal cover of (X, x) compact.

Claim: $\pi_1(X, x)$ finite.

Let $p: \tilde{X} \to X$ be the universal cover.

$p_*(\pi_1(\tilde{X}))$ = trivial subgroup has index equal to the number of sheets in the covering.

Let U be some open set in X.

and let $K \subset U$ be some closed subset.

Then $p^{-1}(K)$ is a closed, and since $\tilde{X}$ compact

$\Rightarrow$ $p^{-1}(K)$ compact. And $p^{-1}(U)$ is an open cover of $p^{-1}(K)$ made of disjoint sets homeomorphic to U. Hence any subcover is the whole cover $\Rightarrow$ finitely sheeted

$\Rightarrow$ index of trivial subgroup of $\pi_1(X)$ is finite $\Rightarrow$ $|\pi_1(X)|$ finite.

Fall 2014 $\boxed{\#2}$

Claim: T^2 and $S^1 \vee S^1 \vee S^2$ have isomorphic homology gps but are not homeomorphic.

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$$

$$\Rightarrow H_n(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases} = H_n(T^2).$$

But they are not homeomorphic because

$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^1 \vee S^1 \vee S^2) = \mathbb{Z} * \mathbb{Z}$

are not equal.

Fall 2014 $\boxed{\#3}$

$f: S^n \to S^n$ continuous.

i) if f has no fixed points $\Rightarrow f \simeq$ antipodal map.

ii) if $n=2m$, $\Rightarrow \exists\, x \in S^{2m}$ s.t. $f(x) = x$ or $f(x) = -x$.

(i) Let $x \in S^n$. Since $f(x) \neq x$, $\exists$ straight line from $-x$ to $f(x)$ that doesn't pass through the origin in $\mathbb{R}^{n+1}$. We then map this straight line to a path γ_x in S^n via the retraction $r: \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ sending $x \longmapsto \frac{x}{|x|}$.

Then $H: S^n \times I \to S^n$ defined by $H(x,t) = \gamma_x(t)$ is a homotopy with $H(x,0) = \gamma_x(0) = -x$ the antipodal map and $H(x,1) = \gamma_x(1) = f(x)$. H is continuous because f is continuous, the straight line homotopy through $\mathbb{R}^{n+1} \setminus \{0\}$ is continuous and r continuous.

(ii) If $\exists$ fixed point, done. If not, (a) $\Rightarrow$ $f \simeq$ antipodal map, α. Then $\deg(f) = \deg(\alpha) = (-1)^{2m+1} = -1$

Then $\deg(\alpha \circ f) = (-1)(-1) = 1 \Rightarrow \alpha \circ f \not\simeq \alpha$

$\overset{(a)}{\Rightarrow}$ $\alpha \circ f$ has a fixed point $\Rightarrow -f(x) = x \Rightarrow f(x) = -x$ for some $x \in S^{2m}$.

2014 Fall $\boxed{\#4}$

See sol'ns.

2014 Fall $\boxed{\#5}$

$X \subset \mathbb{R}^3$ closed submfd homeomorphic to sphere with $g > 1$ handles,

$\Rightarrow \exists$ nonempty open subset on which Gaussian curvature K is negative.

$$\int_X K\,dA = 2\pi\chi(X) = 2\pi(2-2g) < 0$$

by Gauss-Bonnet.

Hence we must have a nonempty open set where $K < 0$.

2014 Fall $\boxed{\# 6}$

M nonempty closed oriented d-dim'l mfd.

ω d-form, X smooth vect. field on M.

Claim: $\mathcal{L}_X \omega$ vanishes at some point on M.

Cartan $\Rightarrow$ $\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)$

$= d(X \lrcorner \omega)$, since $d\omega$ is

a $(d+1)$-form on d-dim'l M and is therefore 0.

$$\int_M \mathcal{L}_X \omega = \int_M d(X \lrcorner \omega) = \int_{\partial M = \emptyset} X \lrcorner \omega = 0$$

Hence $\mathcal{L}_X \omega$ must vanish at some point on M.

2014 Fall $\boxed{\#7}$

Show $\mathbb{C}P^1$ smooth oriented mfd by atlas construction.

Let an element of $\mathbb{C}P^1$ be represented by $[z]$ where $z \in \mathbb{C}^2$. $[z] = [z']$ iff $z = \lambda z'$ for some $\lambda \in \mathbb{C}$. Then Define

$$U_1 = \{[z] \in \mathbb{C}P^1 : z_1 \neq 0\}$$
$$U_2 = \{[z] \in \mathbb{C}P^2 : z_2 \neq 0\}.$$

U_1, U_2 cover $\mathbb{C}P^1$ and are open.

Define $\phi_1 : U_1 \longrightarrow \mathbb{C}$ by $\phi_1([z]) = \frac{z_2}{z_1}$

$\phi_2 : U_2 \longrightarrow \mathbb{C}$ by $\phi_2([z]) = \frac{z_1}{z_2}$

Well defined b/c $\phi_1([\lambda z]) = \frac{\lambda z_2}{\lambda z_1} = \frac{z_2}{z_1} = \phi_1([z])$

Then $\phi_1^{-1} : \phi_1(U_1) \longrightarrow \mathbb{C}P^1$ is defined by

$\phi_1^{-1}(w) = [(1, w)]$. Similarly $\phi_2^{-1}(w) = [(w, 1)]$.

Then $\phi_1^{-1}\left(\frac{z_2}{z_1}\right) = \left[\left(1, \frac{z_2}{z_1}\right)\right] = \left[z_1\left(1, \frac{z_2}{z_1}\right)\right] = [z]$

and $\phi_1([(1,w)]) = \frac{w}{1} = w$. So these are inverses and are continuous, so ϕ_1, ϕ_2 are homeomorphisms onto their images.

We just need to check compatibility

That is,

$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$ must be smooth.

$\phi_2([(1, w)]) = \frac{1}{w}$, $w \neq 0$ b/c we're considering $U_1 \cap U_2$.

This is smooth, so we are done.