2014, Fall

Problem 1.

Let $p : (\tilde{X}, \tilde{x}) \twoheadrightarrow (X, x)$ be the (compact) universal cover of X, where $\tilde{x} \in \tilde{X}$ is some point in the fiber $p^{-1}(x) \subset \tilde{X}$. We have a bijection $\pi_1(X, x) \to p^{-1}(x)$ given by associating to a loop $[f] \in \pi_1(X, x)$ the point $\tilde{f}(1) \in p^{-1}(x)$, where $\tilde{f} : [0,1] \to \tilde{X}$ is the unique lift (of a representative $f : [0,1] \to X$) satisfying $\tilde{f}(0) = \tilde{x}$. But $p^{-1}(x)$ is finite since it's a discrete closed subset of the compact space $\tilde{X}$, and so $\pi_1(X, x)$ must be finite as well. □

Problem 2.

We have that

$$\mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2) \cong \mathsf{H}_j(\mathsf{S}^1) \oplus \mathsf{H}_j(\mathsf{S}^1) \oplus \mathsf{H}_j(\mathsf{S}^2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else} \end{cases} \cong \mathsf{H}_j(\mathsf{T}^2).$$

Yet, $\pi_1(\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2) \cong \mathbb{Z} * \mathbb{Z}$ is nonabelian, while $\pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ is abelian, so $\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2$ and T^2 can't be homeomorphic. □

Problem 3.

Let $a : \mathsf{S}^n \to \mathsf{S}^n$ be the antipodal map given by $a(x) := -x$.

(i) Suppose $f : \mathsf{S}^n \to \mathsf{S}^n$ has no fixed points. Since $\mathsf{S}^n \subset \mathbb{R}^{n+1}$, we may use the vector space structure of $\mathbb{R}^{n+1}$ to define a family of maps $\{h_t : \mathsf{S}^n \to \mathsf{S}^n\}_{0 \le t \le 1}$ by

$$h_t(x) := \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Clearly the denominator is nonzero when $t = 1$ since $x \in \mathsf{S}^n$. Now assume that for some $0 \le t_0 < 1$, the denominator is 0; then $f(x) = \frac{t_0}{1-t_0}x$. Since $\|x\| = \|f(x)\| = 1$, we must have $t_0 = 1/2$. But then $f(x) = x$, a contradiction. Thus the denominator is always nonzero on S^n, whereby $\{h_t\}_{0 \le t \le 1}$ is a well defined homotopy between $h_0 = f$ and $h_1 = a$. □

(ii) With f as above, assume that there are no points $x \in \mathsf{S}^{2m}$ such that $f(x) = x$ or $f(x) = -x$. Then both $-f, f : \mathsf{S}^{2m} \to \mathsf{S}^{2m}$ are free of fixed points, and so by (a) are homotopic to a. Therefore we have $\mathsf{deg}(f) = \mathsf{deg}(a) = (-1)^{2m+1} = -1$, but on the other hand

$$\mathsf{deg}(f) = \mathsf{deg}(a \circ (-f)) = \mathsf{deg}(a)\mathsf{deg}(-f) = (-1)^{2m+1}\mathsf{deg}(-f) = -\mathsf{deg}(a) = -(-1)^{2m+1} = 1,$$

a contradiction. □

Problem 4.

See problem 5 of 2005, Fall, and problem 4 of 2013, Fall.

Problem 5.

We're given that X is homeomorphic to a genus-g surface, and that $\partial X = \varnothing$. So by Gauss-Bonnet,

$$\iint_X K \mathsf{d}A = 2\pi\chi(X) = 2\pi(2 - 2g) < 0$$

since $g > 1$. It follows that $K < 0$ on a subset $U \subset M$ with nonempty interior. Choosing an interior point $x \in U$ and an open neighborhood $V \subset U$ of x, we have that $K < 0$ on V. □

Problem 6.

Since $\omega \in \Omega^d(M)$ is a volume form, then $\mathsf{d}\omega = 0$. Hence by Cartan and Stokes,

$$\int_M \mathcal{L}_X \omega = \int_M (\mathsf{d}\iota_X \omega + \iota_X \underbrace{\mathsf{d}\omega}_{=0}) = \int_{\partial M} \iota_X \omega = 0,$$

because $\partial M = \varnothing$. This implies that $\mathcal{L}_X \omega$ must vanish at some point of M. $\square$

Problem 7.

See problem 3 of 2013, Spring, replacing real numbers by complex ones.