incomplete

Geometry/Topology Qualifying Exam

Fall 2013

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

$S^2 \vee S^1$
cellular
columnar
chain
complex

1. (15 pts) Let X denote S^2 with the north and south poles identified.
   (a) (5 pts) Describe a cell decomposition of X and use it to compute $H_i(X)$ for all $i \geq 0$.
   (b) (5 pts) Compute $\pi_1(X)$.
   (c) (5 pts) Describe (i.e., draw a picture of) the universal cover of X and all other connected covering spaces of X.

open covers. + inv. fct thm.

2. (10 pts) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective.

regular val thm.

3. (10 pts) Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ is a smooth manifold. Here $M_n(\mathbb{R})$ is the set of $n \times n$ real matrices.

do generating $H^1_{dR}(S^1)$

4. (10 pts) Compute the de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

Hatcher

5. (10 pts) Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \ldots$$

?

6. (10 pts) A lens space $L(p, q)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$ for coprime p, q.
   (a) (5 pts) Compute $\pi_1(L(p, q))$.
   (b) (5 pts) Show that any continuous map $L(p, q) \to T^2$ is null-homotopic.

embed in $\mathbb{RP}^2$

7. (10 pts) Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Fall 2013

① $X = S^2/\sim$ north pole $\sim$ s. pole.

(a) Describe a cell decomposition of X & compute $H_i(X)$ $\forall i \geq 0$.

(b) $\pi_1(X)$

(c) Draw univ. cover of X & all other connected covering spaces.

Pf a) $X \simeq$  $\simeq S^2 \vee S^1$ 

so $\Rightarrow$
$\Delta_0 = 1$ zero cell $= \langle v \rangle$
$\Delta_1 = 1$ one cell $\Rightarrow \langle a \rangle$ - glued along v
$\Delta_2 = 1$ two cell $\Rightarrow \langle B \rangle$ - glued along S^1

so $0 \xrightarrow{\partial_3} \Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0$

$\ker \partial_0 = \langle v \rangle$ $\quad$ $\ker \partial_1 = \langle a \rangle$ $\quad$ $\ker \partial_2 = \langle B \rangle$ $\quad$ $\ker \partial_3 = 0$

$\operatorname{im} \partial_0 = 0$ $\quad$ $\operatorname{im} \partial_1 = 0$ $\quad$ $\operatorname{im} \partial_2 = 0$ $\quad$ $\operatorname{im} \partial_3 = 0$

so $H_0 = \dfrac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{Z}$ $\quad$ $H_1 = \dfrac{\ker \partial_1}{\operatorname{im} \partial_2} = \dfrac{\langle a \rangle}{0} = \mathbb{Z}$ ...

$H_2 = \dfrac{\ker \partial_2}{\operatorname{im} \partial_3} = \dfrac{\langle B \rangle}{0} = \mathbb{Z}$ $\Rightarrow$ $H_k(X) = \begin{cases} \mathbb{Z} & \text{for } k = 0,1,2 \\ 0 & \text{else.} \end{cases}$

(b) $\pi_1(X) = \pi_1(S^2) * \pi_1(S^1) = \mathbb{Z}$

(c) the universal cover is an infinite wedge of S^2 & $\mathbb{R}$

all other connected subspace must be compact $\Rightarrow$ bouquets of S^2 & $\mathbb{R}$

$\Rightarrow$  n sheeted $\Rightarrow$ n times since all correspond to $n\mathbb{Z} \leq \mathbb{Z} = \pi_1(X)$.

(2) M compact, N is connected then $f: M \to N$ is surjective for $\forall f$ - submersions.

Pf If f is a submersion $\Rightarrow$ $f_*: T_pM \to T_qN$ is surjective. so then all points in M are regular points so then $f_{*,p} \neq 0$ $\Rightarrow$ by the inverse function theorem $\forall$ $p \in M$ $\exists$ nbd $\ni$ f is a local diffeomorphism. But then let U_p be such nbds $\ni$ $f(U_p) \simeq U_p$. Then M is compact so $f(M)$ is compact $\Rightarrow$ closed. However by compactness $\exists$ finite subcover of points p_i $\ni$ $M = \bigcup^k U_i$ $\Rightarrow$ $f(M) = \bigcup^k f(U_i)$ is open in N since $f(U_i) \simeq U_i$ for each i.. $\Rightarrow$ $f(M)$ is both open & closed in N. Since N is connected $\Rightarrow$ $f(M) = \phi$ or $f(M) = N$. Since $M \neq \phi$ $\Rightarrow$ $f(M) = N$ $\Rightarrow$ f is surjective

(4) Compute de Rham Cohomology of S^1 from defn:

Pf Recall $S^1 \simeq \mathbb{R}^2 - \{pt\}$

so then $H^k_{dR}(S^1) = \dfrac{\ker d: \Omega^k \to \Omega^{k+1}}{\operatorname{Im} d: \Omega^{k-1} \to \Omega^k}$

so $\Omega^0 = \{f \in C^\infty(S^1)\} \Rightarrow \ker d: \Omega^0 \to \Omega^1 = \{f, df = 0\}$.

But locally $S^1 \simeq \mathbb{R}$ so $\Rightarrow f' = 0 \Rightarrow f$ is constant.

Thus $\ker d_0 = \{c;\ c \in \mathbb{R}\} = \mathbb{R}. \Rightarrow H^0_{dR}(S^1) = \mathbb{R}$.

Now, let $w \in \Omega^1(\mathbb{R}^2 - \{p\})$. Then on S^1 $w|_{S^1} = d\theta$ is an orientation form so then $\int_{S^1} w = \int_{S^1} d\theta = 2\pi$.

So then if $\eta \in \Omega^1(\mathbb{R}^2 - \{p\})$ then if $\eta \neq 0 \Rightarrow$ let $c = \frac{1}{2\pi} \int_{S^1} \eta$.

then $\int_{S^1} \eta - cw = 0 \Rightarrow [\eta - cw] = 0 \Rightarrow [\eta] = c[w]$

i.e. $[w]$ generates $H^1_{dR}(S^1) \Rightarrow H^1_{dR}(S^1) = \mathbb{R}$.

...

(5) In Hatcher

(6) ??

⑦ Consider all lines in $\mathbb{R}^2$

Explain how to give it the structure of a smooth manifold.

Is it orientable?

~~Pf~~ any line has the form $ax+by+c=0$ for $a,b,c \in \mathbb{R}$.

Since $\lambda(ax+by+c)=0$ represents the same line $\exists$ natural

map from this set: $S \longrightarrow \mathbb{RP}^2$

$$ax+by+c \longmapsto [a,b,c]$$

In particular, we can consider $\mathcal{U}_a = \{[a,b,c] \, ; \, a \neq 0\}$

$\mathcal{U}_b = \{[a,b,c] \, ; \, b \neq 0\}$ & $\mathcal{U}_c = \{[a,b,c] \, ; \, c \neq 0\}$.

then $\mathcal{U}a = \{[1, b/a, c/a] \, ; \, a \neq 0\}$

Now, $\mathcal{U}a^c = \{[0,b,c]\}$ which is closed in $\mathbb{RP}^2$ by quotient topology.

So $\mathcal{U}a$ is open.

Now let $\varphi_a : \mathcal{U}a \longrightarrow \mathbb{R}^2$ w/ inverse $\varphi_a^{-1}(b,c) \to [1,b,c]$

$$[1, b/a, c/a] \longrightarrow [\tfrac{b}{a}, \tfrac{c}{a}]$$

then $\{(\varphi_i, \mathcal{U}_i)\}$ are a chart for S, hence S is a smooth

manifold. (Clearly 2nd countable & Hausdorff are inherited

from $\mathbb{R}^2$ or $\mathbb{RP}^2$)

Now $\varphi_a^{-1}\varphi_b([1,1,c]) = \varphi_a^{-1}([1,c]) = [1,1,c]$

$\varphi_a^{-1}\varphi_c([1,b,1]) = \varphi_a^{-1}((1,b)) = [1,1,b]$

$\varphi_c^{-1}\varphi_a([1,b,1]) = \varphi_c^{-1}((b,1)) = [b,1,1]$

so $\overset{\text{not}}{\text{i}}$ orientable? $\Rightarrow$ compute transition matrix & find determinant. . . .