Fall 2013 $\boxed{\#1}$

$X = S^2/\{N,S\}$, N, S north, south poles.

(a) Cell decomp of X, compute $H_i(X)$ $\forall i \geq 0$.

(b) $\pi_1(X)$

(c) draw pic of universal cover of X & all other connected covering spaces.

(a) Let e_0 be $N=S$, 0-cell.

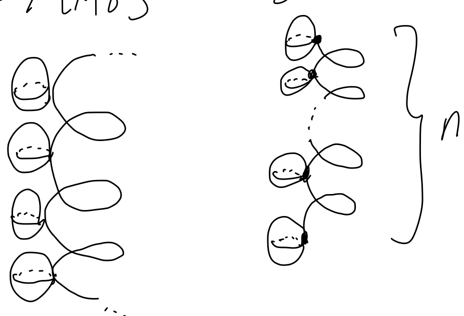
Let e_1 be a meridional line from S to N, 1-cell

Let e_2 be a 2-cell glued along e_1 then back down e_1 to obtain $S^2/\{N,S\}$

$H_2(X) = H_2(X_2, X_1)$

(b) $S^2/\{N,S\} \simeq S^2 \vee S^1 \Rightarrow \pi_1(X) = \mathbb{Z}$.

(c)



Fall 2013 $\boxed{\#2}$

M compact N connected, $f : M \to N$ submersion. Claim: f surjective.

$N = f(M) \sqcup (N - f(M))$. Suppose $N - f(M) \neq \emptyset$.

M compact $\Rightarrow$ $f(M)$ compact $\Rightarrow$ $N - f(M)$ open.

f submersion $\Rightarrow$ $\dim M \geq \dim N$.

Fall 2013 $\boxed{\#3}$

$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ smooth mfd.

Let $f: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$A \longmapsto AA^T$

Then $O(n) = f^{-1}(id)$.

If id is a regular value of f
then $O(n)$ is an embedded smooth mfd.

$df_A(B) = (f \circ \gamma)'(0)$ where

$\gamma(t) = A + tB$

$(f \circ \gamma)(t) = (A + tB)(A + tB)^T = (A + tB)(A^T + tB^T)$

$= AA^T + tBA^T + tAB^T + t^2 BB^T$

$(f \circ \gamma)'(t) = BA^T + AB^T + 2tBB^T$

$(f \circ \gamma)'(0) = BA^T + AB^T$

For any $C \in M_n(\mathbb{R})$, let $B = \frac{1}{2}CA$

Then $df_A(B) = \frac{1}{2}CAA^T + \frac{1}{2}AA^TC = C \Rightarrow df_A$ surjective
$\forall A \in O(n)$.

Fall 2013 $\boxed{\#4}$

Compute de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

$H^n_{dR}(X) =$ closed n-forms / exact n-forms

All 1-forms on S^1 are closed because 2-forms are zero on 1-dim'l S^1.

Let ω be a closed 1-form on S^1.

Assume $\omega = d\alpha$ exact as well.

Then $\int_{S^1} \omega = \int_{S^1} d\alpha = \int_{\partial S^1} \alpha = \int_{\emptyset} \alpha = 0$

Define a map

$$I : H^1_{dR}(X) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_{S^1} \omega$$

Then $\ker(I) = 0 \Rightarrow I$ injective

I surjective b/c $\int_{S^1} \lambda\omega = \lambda \int_{S^1} \omega$.

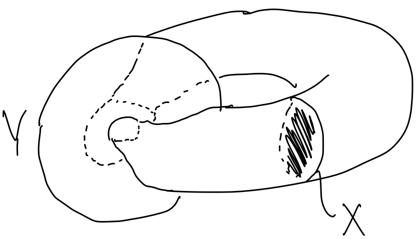
Fall 2013 $\boxed{\# 5}$

$f, g : X \to Y$ continuous. $Z = (X \times [0,1]) \sqcup Y / \sim$

$(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ $\forall\, x \in X$.

Claim: $\exists$ LES

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \cdots$$



$((X \times \{0\} \cup \{1\}) \sqcup Y, Z)$

good pair

Fall 2013 $\boxed{\#6}$

$L(p,q) :=$ quotient of $S^3 \subset \mathbb{C}^2$ by $\mathbb{Z}/p$ action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$, p, q coprime.

(a) Compute $\pi_1(L(p,q))$

(b) any continuous map $L(p,q) \to T^2$ null homotopic

Let $f: S^3 \to L(p,q)$ be the quotient map.

Let $z = (z_1, z_2) \in S^3$. Let $U = \{y \in S^3 : |y - z| < \varepsilon\}$ for ε small enough so that $\{gU : g \in \mathbb{Z}/p\}$ are distinct.

Then $f(U) = f(gU)$ $\forall g \in \mathbb{Z}/p$. And $gU \cong f(U)$ $\forall g \in \mathbb{Z}/p$ because of the subspace topology.

Thus, we have a universal p-sheeted covering space $f: S^3 \to L(p,q)$. Each element $g \in \mathbb{Z}/p$ is a deck transformation since it is just a rotation.

Conversely, let h be a deck transformation, $z \in S^3$.

$\exists$ $g \in \mathbb{Z}/p$ s.t. $g(z) = h(z)$ and deck transforming of S^3 are determined by where a single point is sent b/c S^3 path connected $\Rightarrow$ $g = h$

So $\mathbb{Z}/p = G(S^3)$ group of deck transformations.

Define $\varphi : \pi_1(L(p,q)) \longrightarrow G(S^3) = \mathbb{Z}/p$ by sending
$[\gamma] \longmapsto g$ where

γ is a loop lifting to a path in S^3 from z_0 to z_1 and g is the deck transformation sending z_0 to z_1.

φ is a surjective homomorphism with kernel the loops lifting to loops in S^3, i.e. elts of $f_*(\pi_1(S_3))$ = trivial subgp.

Hence φ is an iso.

$\pi_1(L(p,q)) = \mathbb{Z}/p$

(b)