2013, Fall

Problem 1.

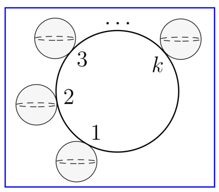
(a) Very similarly to problem 3 of 2006, Spring, we see that $X \cong \mathsf{S}^1 \vee \mathsf{S}^2$. Thus X is obtained by attaching a 1-cell and a 2-cell to a single 0-cell. It's immediate that

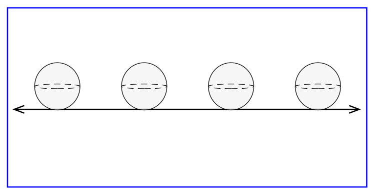
$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, 1, 2, \\ 0 & \text{else.} \end{cases}$$

□

(b) By van Kampen, $\pi_1(X) \cong \pi_1(\mathsf{S}^1) * \pi_1(\mathsf{S}^2) \cong \mathbb{Z} * 1 \cong \mathbb{Z}$. □

(c) Equivalence classes of connected covers of X are in bijection with the subgroups of $\pi_1(X) \cong \mathbb{Z}$. Any proper subgroup of $\mathbb{Z}$ is of the form $k\mathbb{Z}$ for some $k \in \mathbb{N}$, and corresponds to the k-sheeted cover below on the left. The identity subgroup corresponds to the universal cover on the right.





□

Problem 2.

Since $f : M \to N$ is continuous and M is compact, then $\mathsf{im}(f)$ is compact, and in particular closed. Since f is a submersion, then by the implicit function theorem f is locally an open map (a projection), and hence $\mathsf{im}(f)$ is open. But then $\mathsf{im}(f) \subset N$ is a simultaneously closed and open subset of the connected manifold N, and is nonempty since $M \neq \varnothing$. Thus $\mathsf{im}(f) = N$. □

Problem 3.

See problem 4 of 2010, Fall.

Problem 4.

Recall that we have canonical isomorphisms $\Omega^j(\mathsf{S}^1) \cong \mathsf{C}^\infty(\mathsf{S}^1)$ for $j = 0, 1$.

- We know that S^1 is a 1-manifold, so for all $j \neq 0, 1$, we have $\Omega^j(\mathsf{S}^1) = \varnothing$ and hence $\mathsf{H}^j_{\mathsf{dR}}(\mathsf{S}^1) = 0$.
- Thus

$$\mathsf{H}^0_{\mathsf{dR}}(\mathsf{S}^1) \cong \ker(\mathsf{d}^0) \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid \mathsf{d}f = 0\} \cong \{f \in \mathsf{C}^\infty(\mathbb{R}) \mid f \text{ a constant function}\} \cong \mathbb{R}.$$

- Consider the integration map $I : \Omega^1(\mathsf{S}^1) \to \mathbb{R}$ given by $I(\omega) := \int_{\mathsf{S}^1} \omega$. Choosing $\mathsf{d}t \in \Omega^1(\mathsf{S}^1)$ with $\int_{\mathsf{S}^1} \mathsf{d}t = 2\pi$, then any $c \in \mathbb{R}$ has $c = I((c/2\pi)\mathsf{d}t)$, and so $\mathsf{im}(I) = \mathbb{R}$. Moreover it's easily verified that $\ker(I) = \mathsf{im}(\mathsf{d}^0)$, and so

$$\mathsf{H}^1_{\mathsf{dR}}(\mathsf{S}^1) = \frac{\ker(\mathsf{d}^1)}{\mathsf{im}(\mathsf{d}^0)} = \frac{\Omega^1(\mathsf{S}^1)}{\ker(I)} \cong \mathsf{im}(I) = \mathbb{R}.$$

Hence $\mathsf{H}^j_{\mathsf{dR}}(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$ □

Problem 5.

Background. The quotient Z is called the *double mapping cylinder of f and g*. In this problem we form a long exact sequence which relates the homologies of the constituent spaces X and Y to the homology of Z.

Let $\iota : X \times \partial[0,1] \hookrightarrow X \times [0,1]$ be the canonical inclusion, and let $q : (X \times [0,1], X \times \partial[0,1]) \to (Z, Y)$ be the restriction of the given quotient map $(X \times [0,1]) \coprod Y \twoheadrightarrow Z$ to the first component. Then the exact sequences for the relative homology of the good pairs $(X \times [0,1], X \times \partial[0,1])$ and (Z, Y) give, for each $j \in \mathbb{Z}$, the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) & \xrightarrow{\delta} & \mathsf{H}_j(X \times \partial[0,1]) & \xrightarrow{\iota_*} & \mathsf{H}_j(X \times [0,1]) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & \\ \cdots \longrightarrow & \mathsf{H}_{j+1}(Z, Y) & \longrightarrow & \mathsf{H}_j(Y) & \longrightarrow & \mathsf{H}_j(Z) & \longrightarrow \cdots . \end{array}$$

Thus we're done if we can show that $\mathsf{H}_{j+1}(Z, Y) \cong \mathsf{H}_j(X)$ for each $j \in \mathbb{Z}$. To see this, fix some $j \in \mathbb{Z}$, and note that

$$\mathsf{H}_j(X \times \partial[0,1]) \cong \mathsf{H}_j\left((X \times \{0\}) \textstyle\coprod (X \times \{1\})\right) \cong \mathsf{H}_j(X)^{\oplus 2}.$$

Both $X \times \{0\}$ and $X \times \{1\}$ are deformation retracts of $X \times [0,1]$, so ι_* is surjective. Then the outer two maps on the top row are 0, and hence δ is injective. As such,

$$\mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \cong \mathsf{im}(\delta) \cong \mathsf{ker}(\iota_*) \cong \{(\omega, -\omega) \mid \omega \in \mathsf{H}_j(X)\} \cong \mathsf{H}_j(X),$$

so it's enough to show $\mathsf{H}_{j+1}(Z, Y) \cong \mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1])$. Recall that $(X \times [0,1], X \times \partial[0,1])$ and (Z, Y) are good pairs, and q yields induces a homeomorphism $(X \times [0,1])/(X \times \partial[0,1]) \xrightarrow{\sim} Z/Y$. So we may factor the leftmost map q_* as

$$\begin{array}{ccc} \mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) & \xrightarrow{\sim} & \tilde{\mathsf{H}}_{j+1}((X \times [0,1])/(X \times \partial[0,1])) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle \sim} \\ \mathsf{H}_{j+1}(Z, Y) & \xleftarrow{\sim} & \tilde{\mathsf{H}}_{j+1}(Z/Y), \end{array}$$

whereby q_* gives the desired isomorphism $\mathsf{H}_{j+1}(X \times [0,1], X \times \partial[0,1]) \xrightarrow{\sim} \mathsf{H}_{j+1}(Z, Y)$. □

Problem 6.

(a) Observe that $\mathbb{Z}_p$ is a finite group, S^3 is Hausdorff, and the given action of $\mathbb{Z}_p$ on S^3 is free; hence this action is properly discontinuous. Then the canonical quotient map $q : \mathsf{S}^3 \twoheadrightarrow L(p,q)$ is a covering map, and

$$\pi_1(L(p,q)) \cong \pi_1(\mathsf{S}^3/\mathbb{Z}_p) \cong \frac{\pi_1(\mathsf{S}^3/\mathbb{Z}_p)}{q_*(\pi_1(\mathsf{S}^3))} \cong \mathbb{Z}_p,$$

since $\pi_1(\mathsf{S}^3) \cong 1$. □

(b) Denote by $\pi : \mathbb{R}^2 \twoheadrightarrow \mathsf{T}^2$ the universal cover of T^2, and let $f : L(p,q) \to \mathsf{T}^2$ be a continuous map. We have an induced map $f_* : \pi_1(L(p,q)) \to \pi_1(\mathsf{T}^2)$, which must be trivial as $\pi_1(L(p,q)) \cong \mathbb{Z}_p$ has torsion while $\pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ doesn't. Then since $\mathbb{R}^2$ is simply connected, we have that $f_*(\pi_1(L(p,q))) \cong 1 \subset \pi_*(\pi_1(\mathbb{R}^2)) \cong 1$, and so there exists a lift

$$\begin{array}{ccc} & & \mathbb{R}^2 \\ & \overset{\exists\, \tilde{f}}{\nearrow} & \downarrow \pi \\ L(p,q) & \xrightarrow{\ f\ } & \mathsf{T}^2. \end{array}$$

Again since $\mathbb{R}^2$ is simply connected, we may choose a homotopy $\{h_t : L(p,q) \to \mathbb{R}^2\}_{0 \le t \le 1}$ with $h_0 = \tilde{f}$ and $h_1 = c$ for some constant map $c : L(p,q) \to \mathbb{R}^2$. Then $\{\pi \circ h_t\}_{0 \le t \le 1}$ is a homotopy which has $\pi \circ h_0 = \pi \circ \tilde{f} = f$, and $\pi \circ h_1 = \pi \circ c$ (a constant map). □

Problem 7.

- For any $a_1, a_2, a_3 \in \mathbb{R}$, denote by $L_{a_1,a_2,a_3} \subset \mathbb{R}^2$ the line determined by the equation

$$a_1 x + a_2 y + a_3 = 0,$$

  and denote by X the space of all lines in $\mathbb{R}^2$. Note that we can't have $a_1 = a_2 = 0$ while $a_3 \neq 0$, and also that $L_{a_1,a_2,a_3} = L_{ca_1,ca_2,ca_3}$ for any $c \in \mathbb{R} \setminus 0$. Thus we have a well defined inclusion map $\iota : X \hookrightarrow \mathbb{R}\mathsf{P}^2 \setminus \{[0:0:1]\}$ given by $\iota(L_{a_1,a_2,a_3}) := [a_1 : a_2 : a_3]$.

- We equip X with the topology induced by this inclusion, whereby X inherits the Hausdorff and second countable properties of $\mathbb{R}\mathsf{P}^2$. Now set $U_1 := \{[1 : a_2 : a_3] \in X\}$ and consider the homeomorphism $\phi_1 : U_1 \to \mathbb{R}^2$ given by $\phi_1([1 : a_2 : a_3]) := (a_2, a_3)$. (Indeed $U_1 \subset \mathbb{R}\mathsf{P}^2$ is open since its complement $\mathbb{R}\mathsf{P}^2 \setminus U_1$ is closed.) Then (U_1, ϕ_1) is a chart for X, and similarly defining $(U_2, \phi_2), (U_3, \phi_3)$, we obtain an atlas $\{(U_j, \phi_j)\}_{j=1}^3$ for X.

- Next, for any $(a_2, a_3) \in \phi_1(U_1 \cap U_2)$, recalling that $a_2 \neq 0$ on U_2, the transition map

$$\tau := \phi_2 \circ \phi_1^{-1}(a_2, a_3) = \phi_2([1 : a_2 : a_3]) = \phi_2([1/a_2 : 1 : a_3/a_2]) = (1/a_2, a_3/a_2)$$

  is smooth. Similarly the other transition maps are also smooth, so this atlas indeed gives X the structure of 2-manifold.

- Finally observe that, with τ as above,

$$\mathsf{det}(\mathsf{d}\tau_{(a_2,a_3)}) = \begin{vmatrix} \dfrac{\partial \tau_1}{\partial a_2} & \dfrac{\partial \tau_1}{\partial a_3} \\ \dfrac{\partial \tau_2}{\partial a_2} & \dfrac{\partial \tau_2}{\partial a_3} \end{vmatrix} = \begin{vmatrix} -a_2^{-2} & 0 \\ -a_3 a_2^{-2} & a_2^{-1} \end{vmatrix} = -a_2^{-3} < 0.$$

  So τ is orientation-reversing, whereby X is unorientable when equipped with this atlas.

□