Geometry-Topology Qualitfying exam
Fall 2012

Solve all of the problems. Partial credit will be given for partial answers.

1. Denote by $S^1 \subset \mathbf{R}^2$ the unit circle and consider the torus $T^2 = S^1 \times S^1$. Now, define $A \subset T^2 = S^1 \times S^1$ by

$$A = \{(x, y, z, w) \in T^2 \mid (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\ \}.$$

Compute $H^*(T^2, A)$. *Here we regard S^1 as a subset of the plane, hence we indicate points on S^1 as ordered pairs.*

WRONG!

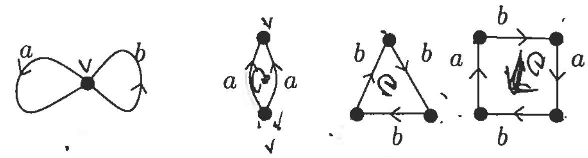
2. Denote by S^1 and S^2 the circle and sphere respectively. Recall that the definition of the smash product $X \wedge Y$ of two pointed spaces is the quotient of $X \times Y$ by $(x, y_0) \sim (x_0, y)$.

mistake?

Show that $S^1 \times S^1$ and $S^1 \wedge S^1 \wedge S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

3. Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.

Cellular homology



1-skeleton 2-skeleton

(a) Compute the homology of X.

(b) Present the fundamental group of X and prove its nonabelian.

(*Justify your work.*)

$\mathbb{RP}^2$ compact + $\mathbb{R}^2$ connected + d_x surj $\Rightarrow \phi$ is surjective

4. Does there exist a smooth embedding of the projective plane $\mathbf{R}P^2$ into $\mathbf{R}^2$? Justify your answer.

5. Let M be a manifold, and let $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbf{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider the vector fields X and Y as maps $C^\infty(M) \to C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$? Explain.

$X = \partial/\partial x$ $\quad Y = \partial/\partial y$
$f = x$ $\quad g = y$

Consider $x \xrightarrow{\phi_t} xf$

Stokes.

6. Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbf{R}^4$. Compute $\int_S \omega$ where $\omega = (w + w^2)dx \wedge dy \wedge dz$.

Regular value thm.

7. Does the equation $x^2 = y^3$ define a smooth submanifold in $\mathbf{R}^3$? Prove your claim.

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① ??

② Mistake? $S^1 \wedge S^1 \wedge S^2 \simeq S^4$

so $H_n(S^4) = \begin{cases} 0 & n \neq 0,4 \\ \mathbb{Z} & n = 0,4 \end{cases}$ but $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\times\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}$

③ Let X be a CW complex w/ one vertex, two one cells & 3 two cells w/

(a) Compute homology of X

(b) Present the fundamental gp & show it is nonabelian

Pf $0 \to \Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0$ $\quad \ker \partial_0 = V \quad \operatorname{im} \partial_0 = 0$

$0 \to \langle U, F, L\rangle \to \langle a, b\rangle \to \langle v\rangle$

$\partial_1(a) = V - V = 0$ $\Rightarrow \operatorname{im}\partial_1 = 0$
$\partial_1(b) = V - V = 0$ $\ldots \ker \partial_1 = \langle a, b\rangle$

$\partial_2(U) = a - a = 0$ $\quad \ker \partial_2 = U$
$\partial_2(F) = b + a + b + a = 2a + 2b$ $\quad \operatorname{im} \partial_2 = \langle 2a+2b, 3b\rangle$
$\partial_2(L) = 3b$

$H_0 = \dfrac{\ker \partial_0}{\operatorname{Im} \partial_1} = \langle v\rangle = \mathbb{Z}$

$H_1 = \dfrac{\ker \partial_1}{\operatorname{im}\partial_2} = \dfrac{\langle a, b\rangle}{\langle 2(a+b), 3b\rangle} = \langle a+b, b \mid 2(a+b) = 3b = 0\rangle = \langle c, b \mid c^2 = b^3 = 0\rangle$
$= \mathbb{Z}_2 \times \mathbb{Z}_3$

$H_2 = \dfrac{\ker \partial_2}{\operatorname{im}\partial_3} = \dfrac{\langle u\rangle}{0} = \mathbb{Z}$

(b) $\pi_1(X) = \langle a, b \mid aa^{-1} = e,\ b^3 = e,\ baba = e \rangle$

$= \langle a, b \mid b^3 = e, (ba)^2 = e \rangle$

$= \langle c, b \mid b^3 = c^2 = e \rangle = \mathbb{F}_2 * \mathbb{F}_3$

(4) Does there $\exists$ a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^2$?

Pf Suppose yes, then if $\phi : \mathbb{RP}^2 \hookrightarrow \mathbb{R}^2$ is a smooth embedding

$\Rightarrow \phi_* : T_p\mathbb{RP}^2 \longrightarrow T_q\mathbb{R}^2$ is injective.

But the dimensions are both $= 2 \Rightarrow \phi_*$ is an isomorphism.

so ϕ_* is surjective. $\Rightarrow \phi$ is a local diffeomorphism.

However $\mathbb{RP}^2$ is compact, & $\mathbb{R}^2$ is connected and ϕ_* surjective $\Rightarrow \phi$ is surjective.

$\Rightarrow \phi(\mathbb{RP}^2) = \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ is compact $\Rightarrow \in$.

So such a map cannot exist.

for justification see #2 Fall 2013

⑤ Let M - manifold, $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbb{R}$.
Explain the relationship btw vector fields on M & $C^\infty(M)$.
If we consider v.f. as maps btw $C^\infty(M) \to C^\infty(M)$ is XY also a v.f? $[X,Y]$?

Pf Given any vector field on M, X, $\phi_X: f \longrightarrow Xf$ defines a smooth vector field on $C^\infty(M)$ iff X is smooth

Where $Xf(p) = X_p f = \left(\sum x_i(p) \frac{\partial}{\partial x_i}\Big|_p\right) f$.

So given XY we want to show it is a derivation.

However $XY(fg) = fXYg + gXYf$

also $XY(fg) = X(Yfg) = Yf\,Xg + g\,XYf$

and $fXYg \neq Yf \cdot Xg$ normally. (see Lee for counterexample)

Now $[X,Y](fg) = (XY - YX)(fg) = XY(fg) - YX(fg)$

$= X(fYg + gYf) - Y(fXg + gXf)$

$= XfYg + XgYf - YfXg - YgXf$

$= fYXg + gXfY + gYXf + \cancel{fXgY} - fXYg - \cancel{gYfX}$

$- gXYf - \cancel{fYgX}$

$= fYXg - fXYg + gYXf - gXYf$

$= f[Y,X]g + g[Y,X]f$ Signs?

⑥ Compute $\int_S \omega = \int (\omega+\omega^2)\, dx \wedge dy \wedge dz$

Pf By stokes.

$$\int_{S^3} \omega = \int_{B^4} d\omega = \int_{B^4} (2w+1)\, dw \wedge dx \wedge dy \wedge dz$$

$$= -\int_{B^4} 2w\, dx \wedge dy \wedge dz \wedge dw - \int_{B^4} dx \wedge dy \wedge dz \wedge dw$$

$$= 0 - \mathrm{vol}(B^4)$$

since B^4 is symmetric & $2w$ is odd $\Rightarrow 2\int_{B^4} \omega = 0$

⑦ is $\{x^2 = y^3\}$ smooth submanifold of $\mathbb{R}^3$?

Pf Consider $S = \{x^2 - y^3 = 0\}$ & let $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$
$(x,y,z) \longmapsto x^2 - y^3$

then $S = \phi^{-1}(0)$

Now $\phi_* : T_p\mathbb{R}^3 \longrightarrow T_{f}\mathbb{R}$ so $\phi_* = [\partial\phi/\partial x \ \partial\phi/\partial y \ \partial\phi/\partial z] = [2x \ -3y^2 \ 0]$

but $\phi_* = 0$ iff $x = y = 0$ so ϕ_* is not surjective at $(0,0)$

since $(0,0) \in S$, then S cannot be a smooth submanifold.