Fall 2012 $\boxed{\#1}$

$S^1 \subset \mathbb{R}^2$, $T^2 = S^1 \times S^1$. $A \subset T^2$

$A = \{(x,y,z,w) \in T^2 : (x,y) = (0,1) \text{ or } (z,w) = (0,1)\}$.

Compute $H^*(T^2, A)$.

Fall 2012 $\boxed{\#2}$

$$X \wedge Y = X \times Y / \sim \quad (x, y_0) \sim (x_0, y)$$

Claim: $H_n(S^1 \times S^1) \approx H_n(S^1 \wedge S^1 \wedge S^2)$

but U.C.'s don't have $\approx$ Homology.

$$S^1 \wedge S^1 = S^1 \times S^1 / \sim \quad (x, y_0) \sim (x_0, y)$$

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

$$(S^1 \wedge S^1) \wedge S^2 = (S^1 \wedge S^1) \times S^2 / \sim \quad ((x_0, y_0), z) \sim ((x, y), z_0)$$

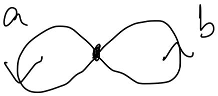
$$= S^1 \times S^1 \times S^2 / \sim \quad (x_0, y, z) \sim (x, y_0, z) \sim (x, y, z_0)$$

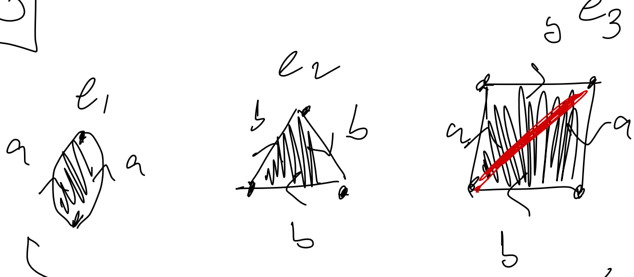
$$= S^1 \times S^1 \times S^2 / \left\{ \{x_0\} \times S^1 \times S^2 \cup S^1 \times \{y_0\} \times S^2 \cup S^1 \times S^1 \times \{z_0\} \right\} =: A$$

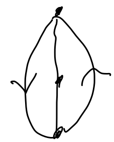
good pair? ✓ $\quad H_n(S^1 \wedge S^1 \wedge S^2) \approx H_n(S^1 \times S^1 \times S^2, A)$

$S^1 \times S^1 \times S^2 = S(SS^2)$ where SX is suspension X.

Fall 2012 $\boxed{\#3}$

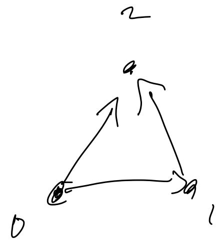






1-skel.

2 skel



(a) Compute $H_n(X)$

(b) $\pi_1(X)$ Claim: nonabelian.

(a) $H_n(X) = 0 \quad \forall\ n \geq 3$

$$C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\times 0} C_0(X) \xrightarrow{\times 0} 0$$

$\partial(e_1) = a - a = 0$

$\partial(e_2) = 3b$

$\partial(e_3) = 2a + 2b$

$$\frac{\langle a, b \rangle}{\langle 2a+2b, 3b \rangle} = \mathbb{Z}$$

(b) $\pi_1(X) = \langle a, b \mid aa^{-1} = b^3 = abab = 1 \rangle$

$= \{0, a, b, b^2, ab, aba, ab^2$

$ba \neq ab$?

If abelian $\Rightarrow$ $abab = aabb = b^2 = 1 \Rightarrow\Leftarrow$

Fall 2012 $\boxed{\#4}$

?$\exists$ smooth embedding of $\mathbb{R}P^2$ into $\mathbb{R}^2$?

Assume so. $f: \mathbb{R}P^2 \longrightarrow \mathbb{R}^2$ smooth embedding

Then $df_p : T_p\mathbb{R}P^2 \longrightarrow T_{f(p)}\mathbb{R}^2 \approx \mathbb{R}^2$ injective

and by dimension surjective.

$f: \mathbb{R}P^2 \longrightarrow f(\mathbb{R}P^2) \subset \mathbb{R}^2$ bijective, so

$f^{-1}: \underset{\mathbb{R}^2}{\underset{\cap}{f(\mathbb{R}P^2)}} \longrightarrow \mathbb{R}P^2$ well defined.

$$\begin{tikzcd} & S^2 \arrow[d, "p"] \\ f(\mathbb{R}P^2) \arrow[ur, dashed, "\tilde{f}^{-1}"] \arrow[r, "f^{-1}"] & \mathbb{R}P^2 \end{tikzcd}$$

$$S^2 \xrightarrow{p} \mathbb{R}P^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow[\cong]{\phi^{-1}} S^2$$

degree 2

Fall 2012 $\boxed{\#5}$

M mfd, $C^\infty(M)$ = algebra of C^∞ functions $M \to \mathbb{R}$. Explain relationship b/w vect. fields on M and $C^\infty(M)$. Consider v.f. X, Y on M as maps $C^\infty(M) \to C^\infty(M)$. Is XY a v.f.? How about $[X,Y] = XY - YX$?

A derivation $v_p : C^\infty(M) \to \mathbb{R}$ satisfies

$v_p(fg) = v(f)g(p) + f(p)v(g)$, linear

Is $(XY)_p$ a derivation $\forall\, p \in M$?

$$(XY)_p(fg) = X_p(Y(fg)) = X_p\big(Y(f)g + fY(g)\big)$$

$$= (XY)_p(f)g(p) + Y(f)(p)X(g)(p)$$

$$+ X(f)(p)Y(g)(p) + f(p)(XY)_p(g)$$

So if $Y(f)(p)X(g)(p) + X(f)(p)Y(g)(p) \neq 0$ for any $p \in M$, then XY is not a vector field.

$$
\begin{aligned}
(XY)_p(fg) - (YX)_p(fg) = {} & XYfg + \cancel{YfXg} + \cancel{XfYg} \\
& + fXYg \\
& - YXfg - \cancel{XfYg} - \cancel{YfXg} \\
& - fYXg
\end{aligned}
$$

$$
= [X,Y](f)\,g + f[X,Y](g) \quad \checkmark
$$

Fall 2012 $\boxed{\#6}$

$S=$ unit sphere in $\mathbb{R}^4$ $\quad x^2+y^2+z^2+w^2=1.$

Compute $\int_S \omega$, $\quad \omega = (w+w^2)\, dx\wedge dy\wedge dz.$

$d\omega = (1+2w)\, dw\wedge dx\wedge dy\wedge dz$. By Stokes

$$\int_S \omega = \int_{B^4} d\omega = \int_{B^4} (1+2w)\, dw\, dx\, dy\, dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^4} 2w\, dw\, dx\, dy\, dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^3}\left(\int_{-\sqrt{1-z^2-y^2-x^2}}^{\sqrt{1-z^2-y^2-x^2}} 2w\, dw\right) dx\, dy\, dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^3} 0\, dx\, dy\, dz$$

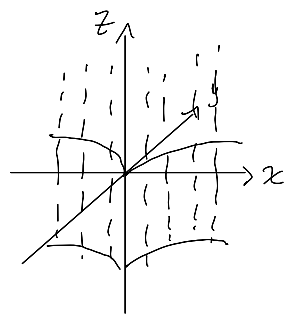
$$= \boxed{-\mathrm{vol}(B^4)}$$

where $B^4 = \{(w,x,y,z)\in\mathbb{R}^4 : x^2+y^2+z^2+w^2\le 1\}$

Fall 2012 $\boxed{\#7}$

$\{x^2 = y^3\} \subset \mathbb{R}^3$ submfd? Call it M.

Assume so. Then



$i : M \hookrightarrow \mathbb{R}^3$ is an immersion.

$(x,y,z) \mapsto (x,y,z)$

Consider the chart $\phi : M \to \mathbb{R}^2$

$(x,y,z) \mapsto (x,z)$

Then $\phi^{-1} : \mathbb{R}^2 \to M$

$(x,z) \mapsto (x, x^{2/3}, z)$

So $i \circ \phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$ call it f

$(x,z) \mapsto (x, x^{2/3}, z)$

Then $df_p = \begin{pmatrix} 1 & 0 \\ \frac{2}{3}x^{-1/3} & 0 \\ 0 & 1 \end{pmatrix}$ is undefined when $x = 0$.

a contradiction.