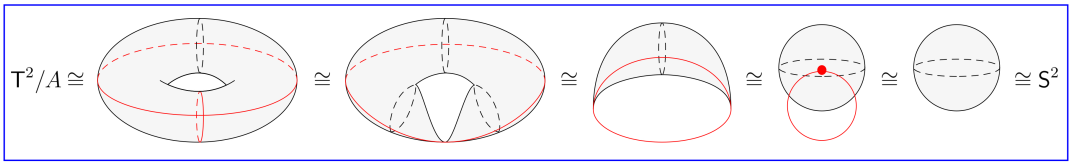
2012, Fall

Problem 1 (?)**.**

Observe that A is the union of a lateral circle and a meridianal one, and that the quotient T^2/A is equivalent to S^2 as shown below.



Hence $\mathsf{H}_j(\mathsf{T}^2/A) \cong \mathsf{H}_j(\mathsf{S}^2)$. Furthermore, A is a deformation retract of a small thickening of itself within T^2. So (T^2, A) is a good pair, whereby $\mathsf{H}^j(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^j(\mathsf{T}^2/A)$ for each $j \geq 0$.

- By properties of reduced cohomology, $\mathsf{H}^0(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^0(\mathsf{T}^2/A) \cong \mathsf{Hom}_{\mathbb{Z}}(\tilde{\mathsf{H}}_0(\mathsf{T}^2/A), \mathbb{Z}) \cong 0$, since $\tilde{\mathsf{H}}_0(\mathsf{T}^2/A) \cong 0$ by path connectedness of T^2/A.
- By Poincaré duality, $\mathsf{H}^1(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^1(\mathsf{T}^2/A) \cong \mathsf{H}_1(\mathsf{T}^2/A) \cong \mathsf{H}_1(\mathsf{S}^2) \cong 0$.
- Similarly, $\mathsf{H}^2(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^2(\mathsf{T}^2/A) \cong \mathsf{H}_0(\mathsf{T}^2/A) \cong \mathsf{H}_0(\mathsf{S}^2) \cong \mathbb{Z}$.
- And finally, for any $j \geq 3$, we have $\mathsf{H}^j(\mathsf{T}^2, A) \cong \tilde{\mathsf{H}}^j(\mathsf{T}^2/A) \cong \tilde{\mathsf{H}}^j(\mathsf{S}^2) \cong 0$.

In summary, $\mathsf{H}^j(\mathsf{T}^2, A) \cong \begin{cases} \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 2.

Remark. This problem's description contains a mistake; the smash product of two (pointed) spaces X, Y is defined by $X \wedge Y := (X \times Y)/(X \vee Y)$. I'm not sure which "definition" of $\wedge$ this problem uses, so this I'll skip this one.

Problem 3.

(a) Recall that the cellular homology of X agrees with its usual singular homology. Let $(\mathsf{C}^{\mathsf{CW}}_\bullet(X), \partial_\bullet)$ denote the cellular chain complex of X,

$$0 \longrightarrow \mathsf{C}^{\mathsf{CW}}_2(X) \xrightarrow{\partial_2} \mathsf{C}^{\mathsf{CW}}_1(X) \xrightarrow{\partial_1} \mathsf{C}^{\mathsf{CW}}_0(X) \xrightarrow{\partial_0} 0$$

and $\mathsf{H}^{\mathsf{CW}}_\bullet(X)$ the homology of this complex. Name the 2-cells A, B, C, in the order that they're pictured.

- By path connectedness, we have $\mathsf{H}^{\mathsf{CW}}_0(X) \cong \mathbb{Z}$.
- We have that $\partial_1(a) = v - v = 0$ and $\partial_1(b) = b - b = 0$, so $\mathsf{ker}(\partial_1) = \mathbb{Z}\langle a, b\rangle$. Next,

$$\partial_2(A) = a - a = 0, \quad \partial_2(B) = 3b, \quad \partial_2(C) = a + b + a + b = 2(a + b),$$

so $\mathsf{im}(\partial_2) = \mathbb{Z}\langle 2(a+b), 3b\rangle$. Observing that $\mathbb{Z}\langle a, b\rangle = \mathbb{Z}\langle a+b, b\rangle$, we have

$$\mathsf{H}^{\mathsf{CW}}_1(X) = \frac{\mathsf{ker}(\partial_1)}{\mathsf{im}(\partial_2)} = \mathbb{Z}\langle a+b, b \mid 2(a+b) = 3b = 0\rangle \cong \mathbb{Z}\langle c, b \mid 2c = 3b = 0\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$

- By the above, $\mathsf{H}_2^{\mathsf{CW}}(X) \cong \ker(\partial_2) = \mathbb{Z}\langle A \rangle \cong \mathbb{Z}$.

Hence $\mathsf{H}_2^{\mathsf{CW}}(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_3 & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

(b) Looking at the 2-skeleton of X, we obtain the presentation

$$\pi_1(X) = \langle a, b \mid aa^{-1} = 1, b^3 = 1, abab = 1 \rangle = \langle ab, b \mid b^3 = 1, (ab)^2 = 1 \rangle \cong \langle d, b \mid d^2 = b^3 = 1 \rangle.$$

The right-hand side is isomorphic to the nonabelian free product $\mathbb{Z}_2 * \mathbb{Z}_3$. □

Problem 4.

Since any embedding is in particular in immersion, the compact 2-manifold $\mathbb{R}\mathsf{P}^2$ can't be embedded into $\mathbb{R}^2$ by problem 1 of 2012, Spring. □

Problem 5.

- Given a vector field $X \in \mathfrak{X}(M)$ and a function $f \in \mathsf{C}^\infty(M)$, we obtain a new function $X(f) \in \mathsf{C}^\infty(M)$ given at each point $x \in M$ by $X(f)(x) := X_x(f)$. In this way, we view X as a map $\mathsf{C}^\infty(M) \to \mathsf{C}^\infty(M)$.
- W.r.t. a local coordinate system $(x_1, \ldots, x_m)$ on the m-manifold M, say $X, Y \in \mathfrak{X}(M)$ are written as $X = \sum_{j=1}^m f_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{j=1}^m g_j \frac{\partial}{\partial x_j}$, for some $f_j, g_j \in \mathsf{C}^\infty(M)$, $1 \leq j \leq m$. Then

$$XY = \sum_{1 \leq i,j \leq m} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{1 \leq i,j \leq m} f_i g_j \frac{\partial}{\partial x_i x_j}$$

is a second-order operator (and hence not a vector field) if the second sum is nonzero.
- However,

$$[X, Y] = XY - YX = \sum_{1 \leq i,j \leq m} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{1 \leq i,j \leq m} g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

is a vector field. Here, the second-order differentials appearing in XY and YX have cancelled by symmetry of mixed partial derivatives.

□

Problem 6.

Note that

$$\int_{\mathsf{S}^3} \omega = \int_{\mathsf{B}^4} \mathsf{d}\omega = \int_{\mathsf{B}^4} (1 + 2w)\mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = \int_{\mathsf{B}^4} \mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z + 2\int_{\mathsf{B}^4} w\mathsf{d}w \wedge \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$$

by Stokes. The second integral on the right vanishes since w is an odd function and B^4 is symmetric about 0. Assuming that $\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w$ is the canonical volume form on $\mathbb{R}^4$, we now have

$$\int_{\mathsf{S}^3} \omega = -\int_{\mathsf{B}^4} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z \wedge \mathsf{d}w = -\mathsf{vol}(\mathsf{B}^4).$$

□

Problem 7.

See problem 3 of 2008, Fall.