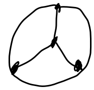
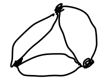
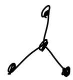
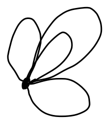
Fall 2010 $\boxed{\#1}$

Compute $\pi_1(X_1), \pi_1(X_2)$ for $X_1 =$  and $X_2 =$ 

These are CW complexes. Let $A =$  which is contractible, and since (X_1, A) good pair we get

$\pi_1(X_1) \approx \pi_1(X_1/A) = \pi_1(S^1 \vee S^1 \vee S^1) = \boxed{\mathbb{Z} * \mathbb{Z} * \mathbb{Z}}$

$B =$  is contractible as well and (X_2, B) good pair.

Then collapsing it gives  $= S^1 \vee S^1 \vee S^1 \vee S^1$

Thus $\pi_1(X_2) \approx \pi_1(X_2/B) = \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \boxed{\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}}$

Fall 2010 $\boxed{\#2}$

Let p_1, p_2, p_3 be three distinct points in S^2

$X = S^2/\{p_1, p_2, p_3\}$. Compute $H_n(X;\mathbb{Z})$.

$S^2/\{p_1,p_2,p_3\}$ is homotopy equivalent to S^2 union two 1-cells glued from p_1 to p_2 and from p_2 to p_3. This is because these two 1-cells are contractible and form a good pair with S^2. Then this is homotopy equivalent with $S^2 \vee S^1 \vee S^1$ by collapsing the paths in S^2 connecting p_1 to p_2 and p_2 to p_3.

Hence $\tilde{H}_n(X) \simeq \tilde{H}_n(S^2 \vee S^1 \vee S^1) = \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$

$$= \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

Thus $$\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}}$$

Fall 2010 $\boxed{\#3}$

Gauss Bonet

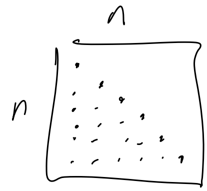
Fall 2010 $\boxed{\#4}$ $\boxed{\text{Daniel}}$

$M_n(\mathbb{R}) = \{n\times n \text{ matrices}\}$. $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = id\}$

Claim: $O(n)$ smooth mfd. wht is it's dimension?

Define $f : M_n(\mathbb{R}) \longrightarrow Sym_n(\mathbb{R})$

$A \longmapsto AA^T$



$df_A : T_A M_n(\mathbb{R}) \longrightarrow T_{AA^T} Sym_n(\mathbb{R})$

$\mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n(n+1)/2}$

$n + n-1 + n-2 + n-3 + \dots + n-(n-1)$

$= n^2 - \sum_{i=1}^{n-1} i$

$= n^2 - \frac{n(n-1)}{2}$

$= \frac{2n^2 - n^2 + n}{2} = \frac{n^2+n}{2}$

$= \frac{n(n+1)}{2}$

$$df_A(B) = \frac{d}{dt}\Big|_{t=0} f(A+tB)$$

$$= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h}$$

$$= \lim_{h\to 0} \frac{(A+hB)(A+hB)^T - AA^T}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{AA^T} + AhB^T + hBA^T + hBhB^T \cancel{-AA^T}}{h}$$

$$= AB^T + BA^T + \cancel{hBB^T}^{\,0}$$

So for $C \in Sym_n(\mathbb{R})$ if $\exists\, B$ s.t. $C = AB^T + BA^T$ then df_A surjective and Id reg. value $\Rightarrow O(n)$ submfd.

Indeed, let $B = \frac{1}{2}CA$. Then $AB^T + BA^T = \frac{1}{2}AA^TC + \frac{1}{2}CAA^T = C$ since $f(A) = Id$. Dimension is then $n^2 - n(n+1)/2 = n(n-1)/2$.

Fall 2010 $\boxed{\#5}$

$\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ $\quad d\omega = dx_1 \wedge \cdots \wedge dx_n$

Claim: $\forall p \in \mathbb{R}$, $\quad \alpha = \frac{1}{(x_1^2 + \cdots + x_n^2)^p} \omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$

is not exact. Hint: S^{n-1}

If α exact, the $\exists\, \eta \in \Omega^{n-2}(\mathbb{R}^n - \{0\})$ s.t. $\alpha = d\eta$.

Then by stokes, $\int_{S^{n-1}} \alpha = \int_{S^{n-1}} d\eta = \int_{\partial S^{n-1}} \eta = 0$

But $\alpha = \omega$ on S^{n-1} $\forall p \in \mathbb{R}$, so by Stokes again

$$\int_{S^{n-1}} \alpha = \int_{S^{n-1}} \omega = \int_{B^n - \{0\}} d\omega = \mathrm{vol}(B^n - \{0\}) \neq 0.$$

Fall 2010 $\boxed{\#6}$

$\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ w/ coords $x_1, y_1, \dots, x_n, y_n$.

$f : \mathbb{R}^{2n} \to \mathbb{R}$ smooth, Find vect. field X s.t.

$i_X \omega = df$ where i_X denotes the interior product. Then

compute Lie derivative $\mathcal{L}_X \omega$.

Recall: $(i_X \omega)_p(v) = \omega_p(X_p, v)$ for $p \in \mathbb{R}^{2n}$, $v \in T_p\mathbb{R}^{2n}$

$$dx_i \wedge dy_i (X_p, v) = \det \begin{pmatrix} x_i(X_p) & x_i(v) \\ y_i(X_p) & y_i(v) \end{pmatrix} = x_i(X_p)\, y_i(v) - x_i(v)\, y_i(X_p)$$

$$\omega_p(X_p, v) = \sum_{i=1}^n x_i(X_p)\, y_i(v) - x_i(v)\, y_i(X_p)$$

$$df_p(v) = \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_p x_i(v) + \left.\frac{\partial f}{\partial y_i}\right|_p y_i(v)$$

Hence $x_i(X) = \dfrac{\partial f}{\partial y_i}$ and $y_i(X) = \dfrac{\partial f}{\partial x_i}$

By Cartan's magic formula,

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega) = X \lrcorner d\omega + \overset{0}{\cancel{d(df)}} = X \lrcorner d\omega$$

$d\omega = 0$ so $X \lrcorner 0 = 0 \Rightarrow \mathcal{L}_X \omega = 0$.

Fall 2010 $\boxed{\#\ 7}$

X top. space. $H_p(X;\mathbb{Z})$ finite. $H^{p+1}(X;\mathbb{Q})=0$.

$u\in C^{p+1}(X;\mathbb{Z})=\mathrm{Hom}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$, $\delta u=0$.

(a) Show $\forall\ \alpha\in C_p(X;\mathbb{Z})$ with $\partial\alpha=0$, $\exists\ k\in\mathbb{Z}-\{0\}$ and $\beta\in C_{p+1}(X;\mathbb{Z})$ with $k\alpha=\partial\beta$.

$[\alpha]\in H_p(X;\mathbb{Z})$, a finite group $\Rightarrow k[\alpha]=0\in H_p(X;\mathbb{Z})$ for some $k\in\mathbb{Z}-\{0\}$. That means $[k\alpha]=0$ $\Rightarrow k\alpha=\partial\beta$ for some $\beta\in C_{p+1}(X;\mathbb{Z})$. $\square$

(b) Show $\exists$ hom $L_u: H_p(X;\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$ s.t. $L_u([\alpha])=\frac{1}{k}u(\beta)$ for every $k\in\mathbb{Z}-\{0\}$ and $\beta\in C_{p+1}(X;\mathbb{Z})$ with $k\alpha=\partial\beta$.

Namely show that $L_u([\alpha])$ independent of k,β and or representative α of $[\alpha]$.

$\delta u=0\Rightarrow[u]\in H^{p+1}(X;\mathbb{Q})=0\Rightarrow\exists\ v\in C^p(X;\mathbb{Z})$ s.t. $\delta v=u$. Thus for β' with $k\alpha=\partial\beta'$ as well, we have

$u(\beta')=\delta v(\beta')=v(\partial\beta')=v(k\alpha)=v(\partial\beta)=\delta v(\beta)=u(\beta)$.

k and k' are the same by order.