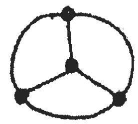
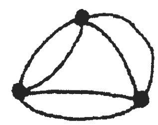
Incomplete
3.

Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$X_1 =$  $X_2 =$ 

Problem 2. Let P_1, P_2, P_3 be three distinct points in the sphere S^2, and let X be the topological space obtained from S^2 by gluing these three points together. Compute all homology groups $H_n(X;\mathbb{Z})$.

0

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface Σ in $\mathbb{R}^3$ at the point p. Does there exist a compact immersed surface Σ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the *orthogonal group* $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = \mathrm{id}\}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $x_1, x_2, \ldots, x_n$ are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p}\,\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: S^{n-1}.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates x_1, $y_1, \ldots, x_n, y_n$. If f is a smooth function on $\mathbb{R}^{2n}$, find the vector field X such that $i_X\omega = df$, where i_X denotes the interior product. Then compute the Lie derivative $\mathcal{L}_X\omega$.

Problem 7. Let X be a topological space such that the homology group $H_p(X;\mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X;\mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k}u(\beta)$$

for every $k \in \mathbb{Z}-\{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$. Namely, show that $L_u([\alpha])$ is independent of k, β and of the representative α of $[\alpha] \in H_p(X;\mathbb{Z})$.

$\partial\alpha = 0$
cycle
$\exists\beta$ s.t. $\partial\beta = \alpha$
boundary.

Check

Q

1. Quotienting by contractible subcomplexes is a homotopy equivalence, so $X_1 \sim S^1 \vee S^1 \vee S^1$ and $X_2 \sim S^1 \vee S^1 \vee S^1 \vee S^1$

2. Same as 1. $X \sim S^2 \vee S^1 \vee S^1$.

3. Product of principal curvatures (intersecting planes containing normal vector) Gauss Bonnet.

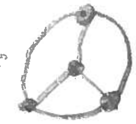
4. Regular Value Theorem

5. Top dimensional De Rham Cohomology isomorphism. Stokes.

6. $i_X(\alpha) = \alpha(X)$ for 1-forms. $i_X(\alpha \wedge \beta) = i_X \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X \beta$.

7. $u = \delta v$ some $v \in C^p(X; \mathbb{Z})$. Just unwind cohomology definitions. $k = |H_p(X; \mathbb{Z})|$.

Geometry/Topology, Fall 2010

① Find $\pi_1(X_1)$ & $\pi_1(X_2)$ where $X_1 =$  , $X_2 =$ 

Recall that the quotient under a contractible subcomplex is homotopy equivalent to original space.

So then: $X_1 \simeq X_1 /$  $\simeq$  $= S^1 \vee S^1 \vee S^1$.

$\Rightarrow \pi_1(X_1) \cong \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

and: $X_2 \simeq X_2 /$  $\simeq$  $\simeq$  $= S^1 \vee S^1 \vee S^1 \vee S^1$

$\Rightarrow \pi_1(X_2) \cong \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

② $P_1, P_2, P_3 \in S^2$, $X = S^2/_{P_1 \sim P_2 \sim P_3}$. Find $H_n(X)$:

$X =$  $\simeq$  $= S^2 \vee S^1 \vee S^1$

So we want $H_n(S^2 \vee S^1 \vee S^1)$; we'll apply Mayer-Vietoris:

Let $A \simeq S^2$, $B \simeq S^1 \vee S^1$, $A \cap B \simeq pt$, $A \cup B = S^2 \vee S^1 \vee S^1$

Then we have:

$$H_n(A) \cong H_n(S^2) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$$ and $H_n(B) \cong H_n(S^1 \vee S^1)$,

hence $H_n(B) = 0$ for $n > 1$, $H_1(B) = \pi_1(B)/[\pi_1(B), \pi_1(B)] = \mathbb{Z} * \mathbb{Z}/(\mathbb{Z} * \mathbb{Z})'$

and $H_0(B) \cong \mathbb{Z}$ since path-connected. $\cong \mathbb{Z} \oplus \mathbb{Z}$

Now we have exact sequence:

$H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B)$

$\to H_1(A \cup B) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$

$\Rightarrow \underline{0 \to \mathbb{Z} \to H_2(X) \to 0} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \underline{H_1(X) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} H_0(X)} \to 0$

$\Rightarrow \underline{H_2(X) \cong \mathbb{Z}}$ by exactness

$\to \mathbb{Z} \oplus \mathbb{Z} = \operatorname{im} \alpha = \ker \beta$

and $\underline{\mathbb{Z} \cong H_0(X)} = \operatorname{im} \delta \Rightarrow \mathbb{Z} = \ker \delta = \operatorname{im} \gamma \Rightarrow 0 = \ker \gamma = \operatorname{im} \beta$

by path-connectedness

$\Rightarrow \underline{H_1(X) = \mathbb{Z} \oplus \mathbb{Z}}$

(can also get H_1 via π_1 ...)

④ Show $O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ smooth mfld. Dimension?

We'll apply regular value theorem:

Consider $f: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$
$$A \longmapsto AA^T$$

Clearly $f^{-1}(I_n) = O_n(\mathbb{R})$, hence we need to show I_n a regular value;

First note that $X \in \operatorname{im} f$ has $X = AA^T \Rightarrow X^T = (AA^T)^T = A^{TT}A^T = AA^T = X$, hence $\operatorname{im} f \subseteq \mathrm{Sym}_n(\mathbb{R})$, so restrict $f: M_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ and choose $A \in f^{-1}(I_n)$ and consider:

$$T_A f : T_A M_n(\mathbb{R}) \to T_{I_n}(\mathrm{Sym}_n(\mathbb{R}))$$

Let $B \in T_A M_n(\mathbb{R})$ and $\gamma(t) = A + tB$. Then:

$$T_A f(B) = (f \circ \gamma)'(0) = \frac{d}{dt}(f(\gamma(t)))\Big|_{t=0} = \frac{d}{dt} f(A+tB)\Big|_{t=0}$$

$$= \frac{d}{dt}\left((A+tB)^T(A+tB)\right)\Big|_{t=0} = \frac{d}{dt}\left(I_n + tA^TB + tB^TA + t^2B^TB\right)\Big|_{t=0}$$

$$= A^TB + B^TA + 2tB^TB\Big|_{t=0} = A^TB + B^TA$$

Recall that $T_{I_n}(\mathrm{Sym}_n(\mathbb{R})) \cong \mathrm{Sym}_n \mathbb{R}$, and if $C \in \mathrm{Sym}_n \mathbb{R}$, we have:

$$C = \frac{1}{2}C + \frac{1}{2}C = \frac{1}{2}C^T + \frac{1}{2}C = \frac{1}{2}C^TA^TA + \frac{1}{2}A^TAC$$

$A^TA = I_n$ since $A \in f^{-1}(I_n)$

$$= \left(\frac{1}{2}C^TA^T\right)A + A^T\left(\frac{1}{2}AC\right)$$

$$= T_Af\left(\frac{1}{2}AC\right)$$

hence $\operatorname{im} T_A f = \mathrm{Sym}_n \mathbb{R}$,

hence $T_A f$ surjective onto $\mathrm{Sym}_n \mathbb{R}$, hence $A \in f^{-1}(I_n)$ regular point, hence I_n reg. value, hence $f^{-1}(I_n) = O_n$ submfld.

Now recall that elements of $\mathrm{Sym}_n\mathbb{R}$ are determined by entries on & above main diagonal, i.e.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ entries, i.e. $\dim \mathrm{Sym}_n\mathbb{R} = \frac{n(n+1)}{2}$

Hence

$$\dim O_n(\mathbb{R}) = \underbrace{\dim M_n(\mathbb{R})}_{\dim(f)} - \underbrace{\dim \mathrm{Sym}_n\mathbb{R}}_{\mathrm{codom}(f)}$$

$$= n^2 - \frac{n(n+1)}{2} = \frac{2n^2 - n^2 - n}{2}$$

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

(5.) $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ s.t. $d\omega = dx_1 \wedge \dots \wedge dx_n$, $p \in \mathbb{R}$.

Show that $\alpha = \dfrac{\omega}{(x_1^2 + \dots + x_n^2)^p} \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ not exact.

Consider α restricted to $S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$

Then we see that $\alpha|_{S^{n-1}} = \omega|_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$.

Then $d\alpha \in \Omega^n(S^{n-1}) = 0$, hence $d\alpha = 0$, hence α closed on S^{n-1}.

Thus we may consider the class $[\alpha] \in H^{n-1}(S^{n-1})$ and apply the top-dim de Rham isomophism:

$$I : H^{n-1}(S^{n-1}) \longrightarrow \mathbb{R}$$
$$[\alpha] \longmapsto \int_{S^{n-1}} \alpha$$

and see that:

Stokes

$$\int_{S^{n-1}} \alpha = \int_{S^{n-1}} \omega = \int_{\partial B^n} \omega = \int_{B^n} d\omega = \int_{B^n} dx_1 \wedge \dots \wedge dx_n$$
$$= \mathrm{vol}(B^n) \neq 0.$$

hence $[\alpha] \neq 0$ in $H^{n-1}(S^{n-1})$, !

hence α restricted to S^{n-1} is <u>not exact</u>.

hence α is not exact on all of $\mathbb{R}^n \setminus \{0\}$.

(6.) $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \in \Omega^2(\mathbb{R}^{2n})$ where $(x_1, y_1, \ldots, x_n, y_n)$ coord. on $\mathbb{R}^{2n}$. If $f: \mathbb{R}^{2n} \to \mathbb{R}$ smooth, find $X \in \mathfrak{X}(\mathbb{R}^{2n})$ such that $i_X \omega = df$, where i_X interior product.
Then find Lie derivative $\mathcal{L}_X \omega$?

The interior product $i_X : \Omega^k(\mathbb{R}^{2n}) \to \Omega^{k-1}(\mathbb{R}^{2n})$ has the properties: (1) for $\omega \in \Omega^1(\mathbb{R}^{2n})$, $i_X(\omega) = \omega(X)$

(2) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X \beta$.

See that $i_X(dx_i \wedge dy_i) = i_X(dx_i) \wedge dy_i - dx_i \wedge i_X dy_i$

$= dx_i(X)\, dy_i - dy_i(X)\, dx_i$

Now let

$$X = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right)$$

and see that $dx_j(X) = \sum_{i=1}^{n} \left(a_i\, dx_j\left(\frac{\partial}{\partial x_i}\right) + b_i\, dx_j\left(\frac{\partial}{\partial y_i}\right) \right)$

$= a_j$ and similarly, $dy_j(X) = b_j$

So we have:

$$i_X(dx_i \wedge dy_i) = a_i\, dy_i - b_i\, dx_i$$
$$= -b_i dx_i + a_i dy_i.$$

Hence $i_X(\omega) = i_X\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \sum_{i=1}^{n} (-b_i dx_i + a_i dy_i)$

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right)$$

hence $\frac{\partial f}{\partial x_i} = -b_i$

$\frac{\partial f}{\partial y_i} = a_i$ $\Rightarrow$ $X = \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial y_i}\right) \frac{\partial}{\partial x_i} - \left(\frac{\partial f}{\partial x_i}\right) \frac{\partial}{\partial y_i} \right]$

Now $X = \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial y_i} \right) \frac{\partial}{\partial x_i} - \left(\frac{\partial f}{\partial x_i} \right) \frac{\partial}{\partial y_i} \right]$

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \in \Omega^2(\mathbb{R}^{2n})$$

and recall $\mathcal{L}_X : \Omega^2(\mathbb{R}^{2n}) \to \Omega^2(\mathbb{R}^{2n})$

$$\omega \longmapsto (d \circ i_X + i_X \circ d)(\omega)$$

So $\mathcal{L}_X(\omega) = d \circ i_X(\omega) + i_X \circ d(\omega)$

$$= d(\cancel{df})^{0} + i_X(d\omega) = i_X(d\omega)$$

$$= \cancel{d\omega(X)}$$ ← only true for ω is 1-form - interior.

But $d\omega = d\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \sum_{i=1}^{n} d(dx_i \wedge dy_i)$

$$= \sum_{i=1}^{n} \underbrace{d(dx_i)}_{\to 0} \wedge dy_i - dx_i \wedge \underbrace{d(dy_i)}_{\to 0}$$

$$= 0$$

hence $\underline{\mathcal{L}_X(\omega) = 0}$.

tangent vector ↔ vector field
same as derivation

$\boxed{\text{\underline{note}: Lie derivative } \mathcal{L}_X : \Omega^k(M) \to \Omega^k(M) \text{ naturally extends derivation } X : \Omega^0(M) \to \Omega^0(M)}$

(7) X space s.t. $|H_p(X,\mathbb{Z})| < \infty$ & $H^{p+1}(X,\mathbb{Q}) = 0$.
Let $u \in C^{p+1}(X,\mathbb{Q}) = \mathrm{Hom}(C_{p+1}(X,\mathbb{Z}),\mathbb{Q})$ cochain with $du = 0$:

(a) Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, $\exists\, k \in \mathbb{Z}\setminus\{0\}$ and $\beta \in C_{p+1}(X,\mathbb{Z})$ with $k\alpha = \partial\beta$.

If $\alpha \in C_p(X)$ has $\partial\alpha = 0$, the homology class $[\alpha] \in H_p(X)$ is well defined.
Now, since $|H_p(X)| < \infty$ it is a torsion $\mathbb{Z}$-module, hence $\exists\, k$ s.t. $k[\alpha] = 0$, namely $k = |H_p(X)|$.
That means that $k[\alpha] = [k\alpha]$ is a boundary, namely $\exists\, \beta \in C_{p+1}(X)$ s.t. $\partial\beta = k\alpha$.

(b) Show $\exists$ hom. $L_u : H_p(X,\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $L_u([\alpha]) = \frac{1}{k}u(\beta)$ for every $k \in \mathbb{Z}\setminus\{0\}$, $\beta \in C_{p+1}(X)$ with $\partial\beta = k\alpha$.

we're given that $du = 0$, hence $[u] \in H^{p+1}(X,\mathbb{Q}) = 0$ is well defined, but clearly $[u] = 0$, hence u is a coboundary, hence $\exists\, v \in C^p(X)$ such that $dv = u$.
Now, since $d = \partial^*$, we have:

$$L_u([\alpha]) = \frac{u(\beta)}{k} = \frac{dv(\beta)}{k} = \frac{v(\partial\beta)}{k} = \frac{v(k\alpha)}{k} = v(\alpha) \in \mathbb{Q}$$

But see that $nk(L_u([\alpha])) = L_u(nk[\alpha]) = L_u(0) = 0$,
hence $L_u([\alpha]) \in \mathbb{Q}/\mathbb{Z}$.