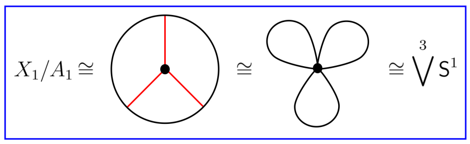
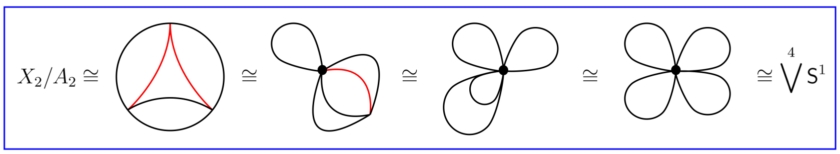
2010, Fall

Problem 1.

If X is a CW complex (for instance, a graph) and $A \subset X$ a contractible subcomplex, then the natural quotient map $X \twoheadrightarrow X/A$ is a homotopy equivalence, whereby $\pi_1(X) \cong \pi_1(X/A)$. We satisfy these assumptions by letting $A_1 \subset X_1$ be the union of the three inner spokes, and $A_2 \subset X_2$ the union of two of the inner segments, as below.





Hence $\pi_1(X_1) \cong \pi_1\left(\bigvee^3 \mathsf{S}^1\right) \cong \mathsf{F}_3$ and $\pi_1(X_2) \cong \pi_1\left(\bigvee^4 \mathsf{S}^1\right) \cong \mathsf{F}_4$. □

Problem 2.

By problem 3 of 2006, Spring, $X \cong \mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^2$. Defining $U \cong \mathsf{S}^1 \vee \mathsf{S}^1, V \cong \mathsf{S}^2$ gives $U \cap V \cong *$, and

$$\mathsf{H}_j(U) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else}, \end{cases} \quad \mathsf{H}_j(V) \cong \begin{cases} \mathbb{Z} & j = 0, 2, \\ 0 & \text{else}. \end{cases}$$

We already know that $\mathsf{H}_0(X) \cong \mathbb{Z}$ since X is path connected. Then by Mayer-Vietoris,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathsf{H}_2(X) \longrightarrow 0 \longrightarrow \mathbb{Z}^{\oplus 2} \xrightarrow{k_1 - \ell_1} \mathsf{H}_1(X) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0, j_0)} \mathbb{Z}^{\oplus 2} \longrightarrow \mathbb{Z}$$

is exact.

- Immediately, $\mathsf{H}_2(X) \cong \mathbb{Z}$.
- By exactness, $\mathsf{ker}(k_1 - \ell_1) \cong 0$, so $\mathsf{ker}(\partial_1) \cong \mathsf{im}(k_1 - \ell_1) \cong \mathbb{Z}^{\oplus 2}$. Next, note that (i_0, j_0) is injective since it's induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of path connected spaces, so $\mathsf{im}(\partial_1) \cong \mathsf{ker}(i_0, j_0) \cong 0$. Thus $\mathsf{H}_1(X) \cong \mathbb{Z}^{\oplus 2}$.

Hence $\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else}. \end{cases}$ □

Problem 3 (?).

No. Suppose $\Sigma \subset \mathbb{R}^3$ is a compact immersed surface without boundary and satisfies $K(x) = -1$ for all $x \in \Sigma$. Then by Gauss-Bonnet,

$$-\mathsf{area}(\Sigma) = -\iint_\Sigma \mathsf{d}A = \iint_\Sigma K\mathsf{d}A = 2\pi\chi(\Sigma) = 2\pi(2-2g),$$

where g is the genus of Σ. Thus $-2\pi(2-2g) = \mathsf{area}(\Sigma) \geq 0$, and so we must have $g \geq 1$. But it's well known that any surface with genus $g \geq 1$ contains points having positive Gaussian curvature, so we've reached a contradiction. □

Problem 4.

Background. The *orthogonal group* $\mathsf{O}(n) \subset \mathsf{Mat}_n(\mathbb{R})$ is the group of isometries of $\mathbb{R}^n$, that is, the group of those matrices $x \in \mathsf{Mat}_n(\mathbb{R})$ which preserve the dot product, $\langle x\cdot, x\cdot\rangle = \langle\cdot,\cdot\rangle$. It's the real counterpart of the *unitary group* $\mathsf{U}(n) \subset \mathsf{Mat}_n(\mathbb{C})$. In this problem we show that $\mathsf{O}(n)$ is a *Lie group.*

Consider the map $f : \mathsf{Mat}_n(\mathbb{R}) \to \mathsf{Sym}(n)$, where $\mathsf{Sym}(n)$ is the space of symmetric $n \times n$ matrices, given by $f(x) := xx^\mathsf{T}$. Then $\mathsf{O}(n) = f^{-1}(1)$. Since f is clearly smooth, we're done if we can show that 1 is a regular value of f. To this end, let $a \in f^{-1}(1)$. Then for any $x \in \mathsf{T}_a\mathsf{Mat}_n(\mathbb{R})$,

$$\begin{aligned}\mathsf{d}f_a(x) &= \lim_{h\to 0}\frac{f(a+hx)-f(a)}{h} = \lim_{h\to 0}\frac{aa^\mathsf{T} + hxa^\mathsf{T} + ahx^\mathsf{T} + h^2xx^\mathsf{T} - 1}{h}\\ &= \lim_{h\to 0}\left(xa^\mathsf{T} + ax^\mathsf{T} + hxx^\mathsf{T}\right) = xa^\mathsf{T} + ax^\mathsf{T}.\end{aligned}$$

The right-hand side is indeed in $\mathsf{Sym}(n)$ since taking its transpose leaves it unchanged. And, $\mathsf{d}f_a$ differential is surjective since for any $y \in \mathsf{Sym}(n)$,

$$\mathsf{d}f_a\left(\frac{1}{2}ya\right) = \frac{1}{2}y\underbrace{aa^\mathsf{T}}_{=1} + \frac{1}{2}\underbrace{aa^\mathsf{T}}_{=1}\underbrace{y^\mathsf{T}}_{=y} = y.$$

This shows that $\mathsf{O}(n)$ is a manifold. To find its dimension, observe that any matrix in $\mathsf{Sym}(n)$ is completely determined by its n diagonal entries and $\frac{1}{2}(n^2-n)$ entries in the upper triangle. So it follows that we have $\dim_\mathbb{R}(\mathsf{Sym}(n)) = n + \frac{1}{2}(n^2-n) = \frac{1}{2}n(n+1)$, and

$$\dim_\mathbb{R}(\mathsf{O}(n)) = \dim_\mathbb{R}(\mathsf{Mat}_n(\mathbb{R})) - \dim_\mathbb{R}(\mathsf{Sym}(n)) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

□

Problem 5.

Note that $\omega = \alpha$ on S^{n-1} since the denominator of α is identically 1 here. Then by Stokes,

$$\int_{\mathsf{S}^{n-1}}\alpha = \int_{\mathsf{S}^{n-1}}\omega = \int_{\mathsf{B}^n}\mathsf{d}\omega = \int_{\mathsf{B}^n}\mathsf{d}x_1\wedge\ldots\wedge\mathsf{d}x_n = \mathsf{vol}(\mathsf{B}^n) \neq 0.$$

If $\alpha = \mathsf{d}\beta$ for some $\beta \in \Omega^{n-2}(\mathbb{R}^n \setminus 0)$, then we obtain the contradiction

$$\int_{\mathsf{S}^{n-1}}\alpha = \int_{\mathsf{S}^{n-1}}\mathsf{d}\beta = \int_{\partial\mathsf{S}^{n-1}}\beta = 0.$$

□

Problem 6.

Suppose $X \in \mathfrak{X}(\mathbb{R}^{2n})$ satisfies $\iota_X \omega = \mathsf{d}f$. Then upon equating the two expressions

$$\iota_X \omega = \sum_{j=1}^{n} \iota_X(\mathsf{d}x_j \wedge \mathsf{d}y_j) = \sum_{j=1}^{n} [(\iota_X \mathsf{d}x_j) \wedge \mathsf{d}y_j - \mathsf{d}x_j \wedge (\iota_X \mathsf{d}y_j)]$$

and

$$\mathsf{d}f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \mathsf{d}x_j + \frac{\partial f}{\partial y_j} \mathsf{d}y_j \right),$$

we have $\mathsf{d}x_j(X) = \iota_X \mathsf{d}x_j = \frac{\partial f}{\partial y_j}$ and $\mathsf{d}y_j(X) = \iota_X \mathsf{d}y_j = -\frac{\partial f}{\partial x_j}$ for each $1 \leq j \leq n$, whereby

$$X = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} \right).$$

Note that $\mathsf{d}\omega = 0$. Then $\mathcal{L}_X \omega = \iota_X \underbrace{\mathsf{d}\omega}_{=0} + \mathsf{d} \underbrace{\iota_X \omega}_{=\mathsf{d}f} = \mathsf{d}(\mathsf{d}f) = 0$ by Cartan. □

Problem 7.

(a) If $\alpha \in \mathsf{C}_p(X;\mathbb{Z})$ has $\partial\alpha = 0$, then α defines a homology class $[\alpha] \in \mathsf{H}_p(X;\mathbb{Z})$. Since $\mathsf{H}_p(X;\mathbb{Z})$ is a finite $\mathbb{Z}$-module, there's some $k \in \mathbb{Z} \setminus 0$ with $k[\alpha] = 0 \in \mathsf{H}_p(X;\mathbb{Z})$, or equivalently, $k\alpha = \partial\beta$ for some $\beta \in \mathsf{C}_{p+1}(X;\mathbb{Z})$. □

(b) The element $u \in \mathsf{C}^{p+1}(X;\mathbb{Z})$ defines a cohomology class $[u] \in \mathsf{H}^{p+1}(X;\mathbb{Q}) \cong 0$ since $\mathsf{d}u = 0$. Then $[u] = 0 \in \mathsf{H}^{p+1}(X;\mathbb{Q})$ and hence $u = \mathsf{d}w$ for some $w \in \mathsf{C}^p(X;\mathbb{Q})$. With α, β, k as above, we define a map $\tilde{L}_u : \mathsf{C}_p(X;\mathbb{Z}) \to \mathbb{Q}$ by

$$\tilde{L}_u(\alpha) := \frac{1}{k} u(\beta) := \frac{1}{k} \mathsf{d}w(\beta) = \frac{1}{k} w(\partial\beta) = \frac{1}{k} w(k\alpha) = w(\alpha).$$

Indeed for any pair β, k satisfying $k\alpha = \beta$, the right-hand side is dependent only on α, so $\tilde{L}_u$ is well defined. Moreover, suppose $\alpha' \in \mathsf{C}_p(X;\mathbb{Z})$ has $[\alpha] = [\alpha'] \in \mathsf{H}_p(X;\mathbb{Z})$. Then $\alpha - \alpha' = \partial\gamma$ for some $\gamma \in \mathsf{C}_{p+1}(X;\mathbb{Z})$, and so

$$\tilde{L}_u(\alpha) - \tilde{L}_u(\alpha') = w(\alpha - \alpha') = w(\partial\gamma) = \mathsf{d}w(\gamma) \in \mathbb{Z} \implies \left[\tilde{L}_u(\alpha)\right] = \left[\tilde{L}_u(\alpha')\right] \in \mathbb{Q}/\mathbb{Z}.$$

Thus we have an induced well defined map $L_u : \mathsf{H}_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ given by $L_u([\alpha]) := [\tilde{L}_u(\alpha)]$. And since $w = \tilde{L}_u$ is a homomorphism, then so is L_u. □