Fall 2009 $\boxed{\#1}$

$f: M \to N$. M, N compact oriented mfds of same dimension n.

$f^*(\pi_1(M)) \subset \pi_1(N)$ has finite index as a subgr.

(a) Prove $[\pi_1(N) : f^*(\pi_1(M))]$ divides the degree of f.

(b) Give ex. where $[\pi_1(N) : f^*(\pi_1(M))] \neq$ degree of f.

(a) We know $H_n(M), H_n(N) = \mathbb{Z}$, so $f_*: H_n(M) \longrightarrow H_n(N)$

$$\mathbb{Z} \to \mathbb{Z}$$
$$x \mapsto kx$$

is multiplication by an integer k which is the degree of x.

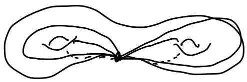
Denote $l := [\pi_1(N) : f^*(\pi_1(M))]$. Let $p: \tilde{N} \to N$ be a covering with $p^*(\pi_1(\tilde{N})) = f^*(\pi_1(M))$, guaranteed to exist by classification of covering spaces. Then the lifting criterion is satisfied and $\exists\ \tilde{f}: M \to \tilde{N}$. Also, since $p^*(\pi_1(\tilde{N}))$ has index l, it is an l-sheeted covering.

$\tilde{N}$ is n-dim'l as well, so we get induced homomorphism on homology $f_* = p_* \circ \tilde{f}_* : H_n(M) \to H_n(N)$.

Hence $\deg(f) = \deg(p)\deg(\tilde{f})$. In particular $\deg(p)$ divides $\deg(f)$, and $\deg(p)$ is l.

(b) $T^2 \to T^2$

$$\pi_1(M_2) \longrightarrow \pi_1(T^2)$$
$$\mathbb{Z}^4 \qquad \mathbb{Z} \times \mathbb{Z}$$



$$\begin{matrix} a \\ b \\ c \\ d \end{matrix} \longmapsto \begin{matrix} a \\ b \\ 0 \\ 0 \end{matrix}$$

$$\begin{tikzcd} & \tilde{N} \arrow[d, "p"] \\ M \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & N \end{tikzcd}$$

ℓ-sheeted

$$\deg(f) = \deg(p \circ \tilde{f}) = \deg(p)\deg(\tilde{f})$$

$$\begin{tikzcd} & H_n(\tilde{N}) \arrow[d, "p^*"] \\ H_n(M) \arrow[ur, "\tilde{f}^*"] \arrow[r, "f^*"'] & H_n(N) \end{tikzcd}$$

$\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}$

?

vlad

Fall 2009 $\boxed{\#2}$

Is there a diff. map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends vect. field $\frac{\partial}{\partial x}$ to v.f. $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and sends v.f. $\frac{\partial}{\partial y}$ to v.f. $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$?

$df_p : T_p\mathbb{R}^2 \longrightarrow T_p\mathbb{R}^2$ Hence, $df_p = \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$ in

$(1,0) \longmapsto (x,1)$

$(0,1) \longmapsto (-1,x)$ matrix form.

Then $\frac{\partial f^1}{\partial x} = x$, $\frac{\partial f^1}{\partial y} = -1$, $\frac{\partial f^2}{\partial x} = 1$, $\frac{\partial f^2}{\partial y} = x$

for $f(x,y) = (f^1(x,y), f^2(x,y))$.

Claim: $\nexists$ $f^2 : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\frac{\partial f^2}{\partial x} = 1$ and $\frac{\partial f^2}{\partial y} = x$

The only f^2 satisfying $\frac{\partial f^2}{\partial y} = x$ are of the form $xy + g(x)$. Then $\frac{\partial f^2}{\partial x} = y + g'(x) \neq 1$.

Hence the answer to the original question is $\boxed{\text{no}}$.

Fall 2009 $\boxed{\#3}$

Let $f: S^n \to S^n$ degree 5.

(a) Show $\exists\ x_1 \in S^n$ s.t. $f(x_1) = -x_1$.

(b) Show $\exists\ x_2 \in S^n$ s.t. $f(x_2) = x_2$.

If $\nexists\ x_1 \in S^n$ s.t. $f(x_1) = -x_1$, then we can draw a line segment from each x to $f(x)$ that doesn't pass through the origin. Let this path be parameterized by $l_x(t)$. Since the line doesn't pass through 0, the path $\dfrac{l_x(t)}{|l_x(t)|}$ is well defined and lies in S^n. These paths give a homotopy from the identity map to f. Then $\deg(f) = \deg(Id) = 1 \Rightarrow\!\Leftarrow$.

This proves (a).

Similarly, if (b) is false then we get a homotopy from f to the antipodal map which has degree ± 1, a contradiction. In this case we draw line segments from $f(x)$ to $-x$.

Fall 2009 $\boxed{\#4}$

M compact submfd of $\mathbb{R}^n$ w/ dim $\leq n-3$.

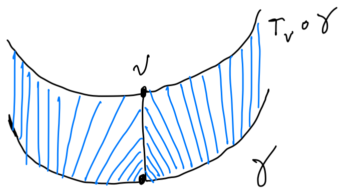
$f: B^2 \to \mathbb{R}^n$ differentiable. $T_v : \mathbb{R}^n \to \mathbb{R}^n$ translation along $v \in \mathbb{R}^n$.

(a) Show $\exists$ arbitrarily small vectors $v \in \mathbb{R}^n$ s.t. the image of $T_v \circ f$ is disjoint from M.

(b) Conclude that $\mathbb{R}^n - M$ simply connected.

First we show (a) $\Rightarrow$ (b).

Let γ be a loop based at x_0 in $\mathbb{R}^n - M$, which is open. $\forall\, x \in \gamma$, there is an open ball nbd containing x in $\mathbb{R}^n - M$, call it $B_{\varepsilon_x}(x)$ where ε_x is the radius. This is an open cover of γ which is compact since $\gamma : [0,1] \to \mathbb{R}^n - M$. Thus, $\exists$ $x_1, \dots, x_k$ finite points s.t. $\gamma \subset \bigcup_{i=1}^{k} B_{\varepsilon_{x_i}}(x_i)$. Let $\varepsilon = \min\{\varepsilon_{x_i}\}_{i=1}^{k}$. Then there is a vector v s.t. $|v| < \varepsilon$ and $(T_v \circ f)(B^2)$ disjoint from M. Now, we can build a homotopy from γ to the constant map C_{x_0}. First there is a homotopy from γ to $\tilde{\gamma} := \nu \cdot T_v \circ \gamma \cdot \nu^{-1}$ where ν is the path from x_0 to $x_0 + v$. We can see that in this diagram:



Then $\tilde{\gamma}$ is homotopic to C_{x_0} since $T_v \circ \gamma \simeq C_{x_0+v}$ with $T_v \circ f$ serving as a homotopy.

Since γ was arbitrary, $\pi_1(\mathbb{R}^n - M)$ trivial.

Now we prove (a).

Let $g: M \times B^2 \longrightarrow \mathbb{R}^n$ be smooth. $\dim(M \times B^2) < n$ so by Sard's thm, $g(M \times B^2)$ has measure 0 in $\mathbb{R}^n$.

if we can have $g(M \times B^2) = \{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}$ then we are done because if (a) is false then $\{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}$ would have positive measure.

Define $g(x, y) = x - f(y) \in \mathbb{R}^n$ for $x \in M \subset \mathbb{R}^n$, $y \in B^2$.

Then
$$\begin{aligned} g(M \times B^2) &= \{v \in \mathbb{R}^n : v = x - f(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n ; x = v + f(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n : x = (T_v \circ f)(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}. \end{aligned}$$

Fall 2009 $\boxed{\#5}$

$\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ closed. $f, g : S^n \to \mathbb{R}^{n+1} - \{0\}$ diff.

Claim: the ratio $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} \in \mathbb{Q}$ when denominator $\neq 0$.

Define $r : \mathbb{R}^{n+1} - \{0\} \to S^n$ by $r(x) = \frac{x}{|x|}$, a retraction.

Then $r \circ f, r \circ g : S^n \to S^n$ have degrees K_f and $K_g \in \mathbb{Z}$.

That is, $\int_{S^n} (r \circ f)^* \alpha = K_f \int_{S^n} \alpha$ for α any diff. form. on S^n.

Is there an $\alpha \in \Omega^n(S^n)$ s.t. $r^*(\alpha) = \omega$?

$i : S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$ is the inclusion map.

Then $r \circ i = Id \Rightarrow i^* \circ r^* = (r \circ i)^* = Id^* = Id$

$\Rightarrow r^*$ surjective, so yes, $\exists\, \alpha$ s.t. $r^*(\alpha) = \omega$.

Then $(r \circ f)^* \alpha = f^*(r^*(\alpha)) = f^*(\omega)$ and $(r \circ g)^* = g^*(\omega)$.

Then

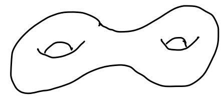
$\int_{S^n} f^*(\omega) = K_f \int_{S^n} \alpha$ and $\int_{S^n} g^*(\omega) = K_g \int_{S^n} \alpha$

Hence, $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} = \dfrac{K_f}{K_g} \in \mathbb{Q}$ when $K_g \neq 0$.

Fall 2009 $\boxed{\#6}$

$S =$ surface of genus $2 \in \mathbb{R}^3$. W is the closure of the bounded component of $\mathbb{R}^3 - S$, i.e. the solid with boundary S, $W \simeq S^1 \vee S^1$. Compute $H_n(W, S)$.

We get the relative long exact sequence of homology:

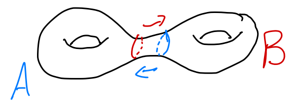


$$\cdots \longrightarrow H_n(S) \longrightarrow H_n(W) \longrightarrow H_n(W,S) \longrightarrow \cdots$$

$$\tilde{H}_n(W) \approx \tilde{H}_n(S^1 \vee S^1) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

So $H_n(W) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$

Let's do M.V. for S. Let $A =$ left torus and $B =$ right torus as such:



Then $A \cap B \simeq S^1$, $A, B = T^2 - \{*\} \simeq S^1 \vee S^1$

So we get:

$$\cdots \longrightarrow \underbrace{H_n(S^1)}_{\begin{cases} \mathbb{Z} & n=0,1 \\ 0 & \text{else} \end{cases}} \longrightarrow \underbrace{H_n(S^1 \vee S^1) \oplus H_n(S^1 \vee S^1)}_{\begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else} \end{cases}} \longrightarrow H_n(S) \longrightarrow \cdots$$

For $n \geq 3$ we have $0 \to H_n(S) \to 0 \Rightarrow H_n(S) = 0$.

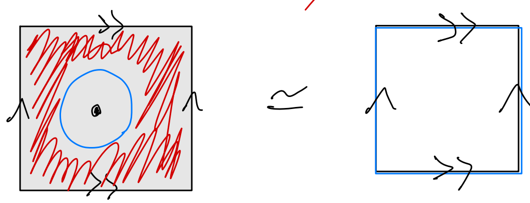
Otherwise, we have (since $H_0(S) = \mathbb{Z}$ by path connectedness)

$$0 \to H_2(S) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(S) \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$$

Consider $H_1(S^1) \xrightarrow{i_*, j_*} H_1(S^1 \vee S^1) \oplus H_1(S^1 \vee S^1)$

$$\mathbb{Z} \longrightarrow \mathbb{Z}^4$$

If we have a loop in $A \cap B$, then we have a loop around the puncture in A. ~~As we can see in the square representation of the torus, the loop gets sent to $aba^{-1}b^{-1}$ as $T - \{*\}$ gets sent to $S^1 \vee S^1$,~~ And $aba^{-1}b^{-1} = 0$ in homology



this is a boundary in the torus of red cell so 0 in homology.

Hence $\mathbb{Z} \to \mathbb{Z}^4$ is the 0-map, and we have

$$0 \xrightarrow{\times 0} H_2(S) \longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}^4$$

implying $H_2(S) \approx \mathbb{Z}$ by exactness.

The map $H_0(S^1) \longrightarrow H_0(A) \oplus H_0(B)$ has kernel 0

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
$$x \longmapsto (a, b)$$

so $\operatorname{im}(H_1(S) \to \mathbb{Z}) = 0$.

Then we have $\xrightarrow{\times 0} \mathbb{Z}^4 \to H_1(S) \xrightarrow{\times 0}$ implying $H_1(S) \approx \mathbb{Z}^4$ by exactness. Hence
$$H_n(S) \simeq \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}$$

Then our original sequence is

$$\cdots \longrightarrow H_n(S) \longrightarrow H_n(W) \longrightarrow H_n(W, S) \longrightarrow \cdots$$

For $n \geq 4$ we have $0 \to H_n(W,S) \to 0 \Rightarrow \boxed{H_n(W,S) = 0.}$

For $n = 3$ we have $0 \to H_3(W,S) \to \mathbb{Z} \to 0 \Rightarrow \boxed{H_3(W,S) = \mathbb{Z}}.$

otherwise we have

$$0 \to H_2(W,S) \to \mathbb{Z}^4 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(W,S) \to \mathbb{Z} \to \mathbb{Z} \to H_0(W,S) \to 0$$

The final $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, so $\boxed{H_0(W,S) = 0}$ since $H_0(W,S) \hookrightarrow 0$. This also means $\operatorname{im}(\mathbb{Z} \oplus \mathbb{Z} \to H_1(W,S))$
$= \ker(H_1(W,S) \xrightarrow{\times 0} \mathbb{Z}) = H_1(W,S)$.

Consider
$$\begin{array}{ccc} H_1(S) & \longrightarrow & H_1(W) \\ \mathbb{Z}^4 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ \left.\begin{array}{c} a \\ b \\ c \\ d \end{array}\right\} & \longmapsto & \left\{\begin{array}{c} x \\ 0 \\ y \\ 0 \end{array}\right. \end{array}$$
which has $\ker = \mathbb{Z} \oplus \mathbb{Z}$ and image $= \mathbb{Z} \oplus \mathbb{Z}$

Then $\operatorname{im}(H_2(W,S) \hookrightarrow \mathbb{Z}^4) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \boxed{H_2(W,S) = \mathbb{Z} \oplus \mathbb{Z}}$

and $\ker(\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow H_1(W,S)) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow$ image $= \boxed{0 = H_1(W,S)}$

In Sum:
$$\boxed{H_n(W,S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z} & n = 3 \\ 0 & \text{else} \end{cases}}$$

Fall 2009 $\boxed{\#7}$

M compact, connected submfd of oriented mfd N of dim n
dim M = dim $N - 1$. Claim: M orientable $\Leftrightarrow$ it admits arbitrarily small connected nbds U s.t. $U - M$ is disconnected.
I.e. $\Leftrightarrow$ $\forall$ open $V \subset N$ containing M, $\exists$ open connected $U \subset V$ s.t. $U - M$ not connected. $M \subset U$?

Let η be an orientation form on N. i.e. η nonvanishing n-form.
Let M be orientable with ω orientation form.
Since M is compact, it is covered by finitely many, positively oriented coordinate charts $U^1, \dots, U^k$ with positively oriented coordinate frames $(E_i^1), \dots, (E_i^k)$,
Then we get a positive direction at each point based on the orientation of N, which is consistent.

? Ask