1. Lifting Criterion, Multiplicativity of degrees, $p: \tilde{N} \to N$ finite sheeted cover and N compact $\Rightarrow$ $\tilde{N}$ compact.

2 f acts on a vector space by its tangent map.

3. If f fixes no point, f is homotopic to the antipode which has degree ± 1. If f sends no point to its antipode, $f \sim id$ so $\deg f = 1$.

4. Sard (Consider $F: M \times B^2 \to \mathbb{R}^n$ $F(x,y) = x - f(y)$)

5. Degree (Must define a retraction from $\mathbb{R}^{n+1} - \{0\}$ to S^n }.

6. Relative Homology Long Exact Sequence.

7. Tubular neighborhoods ($T(M) \cong M \times I$ if M is oriented, so $T(M) - M \cong M \times (-\varepsilon, 0) \cup (0, \varepsilon]$)

Geometry / Topology - Fall 09 :

1. $f: M \to N$ map between compact, oriented manifolds, $\dim M = \dim N$.
Suppose $f^*(\pi_1(M)) \leq \pi_1(N)$ has finite index. So degree is well-defined. 10

(a) Show that $[\pi_1(N) : f^*(\pi_1(M))] \mid \deg f$.

Let $p: \tilde{N} \to N$ be the cover corresponding to the subgroup $f^*(\pi_1(M)) \leq \pi_1(N)$. By construction, the number of sheets of p is $[\pi_1(N) : f^*(\pi_1(M))]$, i.e. $\deg p = [\pi_1(N) : f^*(\pi_1(M))] < \infty$

Because of our construction we have $f_*(\pi_1(M)) = p_*(\pi_1(\tilde{N}))$, hence the lifting criterion is satisfied, and we therefore have $\tilde{f}: M \to \tilde{N}$ such that $f = p \circ \tilde{f}$.

If $\deg f = 0$, statement is trivial, so assume $\deg f \neq 0$.
Then $\deg f = \deg p \circ \tilde{f} = \deg p \cdot \deg \tilde{f} \Rightarrow [\pi_1(N) : f^*(\pi_1(M))] = \deg p \mid \deg f$

$\underbrace{[\pi_1(N) : f^*(\pi_1(M))]}_{\text{no. of sheets of } \tilde{N}}$

(b) Example where $[\pi_1(N) : f^*(\pi_1(M))] < \deg f$

(2) Is there a differentiable map relating $\frac{\partial}{\partial x}$ & X, $\frac{\partial}{\partial y}$ & Y where $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ & $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

By the naturality of the Lie bracket, if $\frac{\partial}{\partial x}$ & X, $\frac{\partial}{\partial y}$ & Y are F-related, then $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$ & $[X, Y]$ are F-related.

Now, see that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ and:

$$[X,Y] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i} \quad \text{where } X_1 = x,\ X_2 = 1,\ Y_1 = -1,\ Y_2 = x$$

$$= \left(X^1\frac{\partial Y^1}{\partial x} - Y^1\frac{\partial X^1}{\partial x} + X^2\frac{\partial Y^1}{\partial y} - Y^2\frac{\partial X^1}{\partial y}\right)\frac{\partial}{\partial x}$$

$$+ \left(X^1\frac{\partial Y^2}{\partial x} - Y^1\frac{\partial X^2}{\partial x} + X^2\frac{\partial Y^2}{\partial y} - Y^2\frac{\partial X^2}{\partial y}\right)\frac{\partial}{\partial y}$$

$$= (x\cdot 0 + (-1)(1) + 0 - 0)\frac{\partial}{\partial x} + (x\cdot(1) - (-1)(0) + 0 - 0)\frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \neq 0,$$

hence $[X,Y]$ & $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$ cannot be F-related,

hence neither may their constituents.

3) $f: S^n \to S^n$ degree 5

(a) show $\exists x_1 \in S^n$ such that $f(x_1) = -x_1$

(b) show $\exists x_2 \in S^n$ such that $f(x_2) = x_2$

(a) Suppose $f: S^n \to S^n$ sends no point to its antipode, i.e. $\nexists x \in S^n$ such that $f(x) = \alpha(x)$, i.e. $\alpha \circ f(x) \neq x$ for all $x \in S^n$.

Since $\alpha \circ f: S^n \to S^n$ has no fixed points, it is homotopic to the antipodal map $\alpha: S^n \to S^n$, hence $(-1)^{n+1} = \deg \alpha = \deg \alpha \circ f = \deg \alpha \cdot \deg f$

$\Rightarrow \deg f = 1$ ↯ contradiction since $\deg f = 5$

(b) Suppose $f: S^n \to S^n$ has no fixed points.
Then f is homotopic to the antipodal map, hence $\deg f = \deg \alpha = (-1)^{n+1}$ ↯ contradiction since $\deg f = 5$

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4) $M \subseteq \mathbb{R}^n$ cpt submfld, $\dim M \leq n-3$, $f: B^2 \to \mathbb{R}^n$ diff map, $T_v: \mathbb{R}^n \to \mathbb{R}^n$, $w \mapsto w+v$, translation along $v \in \mathbb{R}^n$.

(a) Show $\exists$ arbitrarily small $v \in \mathbb{R}^n$ s.t. $T_v(\operatorname{im} f) \cap M = \phi$

(b) Show $\mathbb{R}^n \setminus M$ is simply-connected.

(a) Consider the set $\{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq \phi\}$

Now see that: $\{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq 0\} =$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ st. } T_v(f(y)) = x\}$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ st. } f(y) + v = x\}$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ s.t. } v = x - f(y)\}$ $= \operatorname{im} F$

Now define the map $F: M \times B^2 \to \mathbb{R}^n$, $(x, y) \mapsto x - f(y)$, hence ~~we~~ have

Then see that $T_pF: T_p(M \times B^2) \to T_p\mathbb{R}^n$ is a map from an at most $(n-3)+2 = n-1$ dimensional space to an n-dim space, hence cannot be surjective, hence $\operatorname{im} F$ consists of critical values, hence $\operatorname{im} F$ measure 0 by Sard

$\Rightarrow \{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq 0\}$ measure 0

$\Rightarrow \{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M = 0\}$ full measure $\Rightarrow v$ can be arb. small

4 cont'd:

(b) By part (a), for any differentiable map $f: B^2 \to \mathbb{R}^n$
we can get $\mathrm{im}(T_v \circ f) \cap M = \emptyset$ for arbitrarily small v.

Consider the restriction $f: S^1 \to \mathbb{R}^n$, which is a loop in $\mathbb{R}^n$
By part (a), we may translate by small v and get
a loop in $\mathbb{R}^n \setminus M$, i.e. T_v of a loop in $\mathbb{R}^n \setminus M$.
But we have actually translated all of $\mathrm{im} f = f(B^2)$.
i.e. the interior of the loop is also in $\mathbb{R}^n \setminus M$, hence
it is nullhomotopic

Since every loop is just a map $f: S^1 \to \mathbb{R}^n$
we have they are all nullhomotopic $\Rightarrow \underline{\pi_1(M \setminus \mathbb{R}^n) = 1}$

(5) $\omega \in \Omega^n(\mathbb{R}^{n+1}\setminus\{0\})$ closed.
Show that, for any two differentiable maps $f, g: S^n \to \mathbb{R}^{n+1}\setminus\{0\}$, the ratio: $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}$ is rational when the denom $\neq 0$.

Let $r: \mathbb{R}^{n+1}\setminus\{0\} \to S^n$, $x \longmapsto \frac{x}{\|x\|}$ be the deformation retraction of $\mathbb{R}^{n+1}\setminus\{0\}$ onto S^n. Then $r\circ f, r\circ g: S^n \to S^n$, hence degree is well-defined, i.e.:

$$(r\circ f)^*: H^n(S^n) \to H^n(S^n), \quad [\omega] \longmapsto a[\omega] \quad \& \quad (r\circ g)^*: H^n(S^n) \to H^n(S^n), \quad [\omega] \longmapsto b[\omega]$$

where a and b are integers.
Furthermore, we have that $(r\circ f)^*(\omega) = r^* \circ f^*(\omega) = a\omega$
$(r\circ g)^*(\omega) = r^* \circ g^*(\omega) = b\omega$

Recall that since r is deformation retraction, r^* is isomorphism; i.e. it is invertible; hence:

$$f^*(\omega) = (r^*)^{-1}(a\omega) = a(r^*)^{-1}(\omega)$$
$$g^*(\omega) = (r^*)^{-1}(b\omega) = b\cdot(r^*)^{-1}(\omega)$$

since homomorphism, and then:

$$\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} = \frac{\int_{S^n} a(r^*)^{-1}(\omega)}{\int_{S^n} b(r^*)^{-1}(\omega)} = \frac{a\cancel{\int_{S^n}(r^*)^{-1}(\omega)}}{b\cancel{\int_{S^n}(r^*)^{-1}(\omega)}} = \frac{a}{b}$$

a rational number since $a, b \in \mathbb{Z}$.

⑥ Let S be the surface of genus 2 and let W be the solid version. Compute $H_n(W,S)$

$S =$  , $W =$  solid

Recall the Long Exact Sequence for Relative Homology:

$$0 \to H_3(S) \to H_3(W) \to H_3(W,S) \to H_2(S) \to H_2(W) \to H_2(W,S)$$
$$\to H_1(S) \to H_1(W) \to H_1(W,S) \to H_0(S) \to H_0(W) \to H_0(W,S) \to 0$$

now, see that $W =$ solid double torus $\simeq \infty = S^1 \vee S^1$, therefore:

$$H_n(W) \cong H_n(S^1 \vee S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$$

and S is the surface of genus 2, hence:

$$H_n(S) \cong H_n(M_2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$$

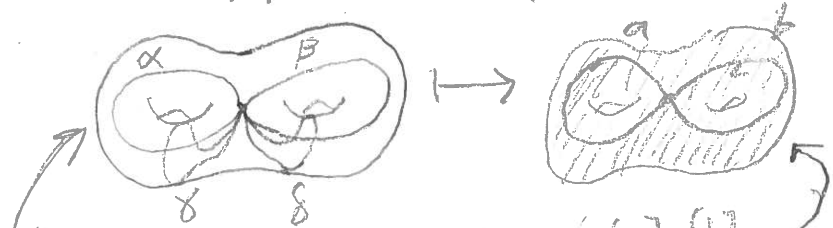
So our sequence reduces to:

$$\underline{0 \to H_3(W,S) \xrightarrow{\partial_3} \mathbb{Z} \to 0} \to H_2(W,S) \xrightarrow{\partial_2} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}\oplus\mathbb{Z}$$
$$\to H_1(W,S) \xrightarrow{\partial_1} \mathbb{Z} \to \mathbb{Z} \to H_0(W,S) \xrightarrow{\partial_0} 0$$

By exactness, $\underline{H_3(W,S) \cong \mathbb{Z}}$.

Now consider the induced map of inclusion on the first homology groups:

$$H_1(S) \xrightarrow{i_*} H_1(W)$$



$\alpha,\beta,\gamma,\delta$ the generators of $H_1(S)$

$[a],[b]$ generators of $H_1(W)$

see that i sends
$\alpha \mapsto a$
$\beta \mapsto b$
and γ and δ to nullhomotopic loops (these circles are included in solid double-torus)
$\Rightarrow \gamma \mapsto 1, \beta \mapsto 1$

Therefore we see that $\ker i_* \cong \mathbb{Z}\oplus\mathbb{Z}$ and $\operatorname{im} i_* \cong \mathbb{Z}\oplus\mathbb{Z}$

Now, by exactness, we see from $0 \to H_2(W,S) \xrightarrow{\partial_2} H_1(S)$ that ∂_2 is injective, hence $H_2(W,S) \cong \operatorname{im}\partial_2 \cong \ker i_* \cong \mathbb{Z}\oplus\mathbb{Z}$

$\Rightarrow \underline{H_2(W,S) \cong \mathbb{Z}\oplus\mathbb{Z}}$

Now consider $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{j_*} H_1(W,S) \xrightarrow{\partial_1} \underset{\overset{\|}{H_0(S)}}{\mathbb{Z}} \xrightarrow{i_*} \underset{\overset{\|}{H_0(W)}}{\mathbb{Z}}$

We have now $\mathbb{Z}\oplus\mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_* \Rightarrow 0 \cong \operatorname{im} j_* \cong \ker \partial_1$
$\Rightarrow \partial_1$ injective,
hence $H_1(W,S) \cong \operatorname{im}\partial$

Now, the inclusion map induced on the 0^{th} homology groups must be injective since both are path-connected, hence $\ker(H_0(S) \xrightarrow{i_*} H_0(W)) \cong 0$, i.e. $H_1(W,S) \cong \operatorname{im}\partial \cong \ker i_* \cong 0$, so we have $\underline{H_1(W,S) \cong 0}$

Finally, we have:

$H_1(W,S) = 0 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} H_0(W,S) \to 0$,

which has i_* injective, j_* surjective by exactness.

See that i_* inj. $\Rightarrow \mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_*$
$\Rightarrow \operatorname{im} j_* = 0 \Rightarrow \underline{H_0(W,S) \cong 0}$ by surjectivity of j_*

$$\text{So: } H_n(W,S) \cong \begin{cases} 0 & n > 3 \\ \mathbb{Z} & n = 3 \\ \mathbb{Z}\oplus\mathbb{Z} & n = 2 \\ 0 & n = 1 \\ 0 & n = 0 \end{cases}$$