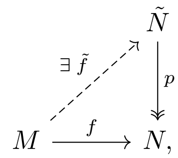
2009, Fall

Problem 1.

(a) Let $p : \tilde{N} \twoheadrightarrow N$ be the cover corresponding to the subgroup $f_*(\pi_1(M)) \subset \pi_1(N)$. This cover has k sheets, where $k := \mathsf{deg}(p) = [\pi_1(N) : f_*(\pi_1(M))]$; note that k is a finite integer by assumption, and is nonzero since p is a covering map. By definition of p, there exists a lift



whereby $\mathsf{deg}(f) = \mathsf{deg}(p \circ \tilde{f}) = \mathsf{deg}(p)\mathsf{deg}(\tilde{f}) = [\pi_1(N) : f_*(\pi_1(M))]\mathsf{deg}(\tilde{f})$. □

(b) The antipodal map $a : \mathsf{S}^2 \to \mathsf{S}^2$ given by $a(x) := -x$ has $\mathsf{deg}(a) = -1$, but since S^2 is simply connected, we have $[\pi_1(\mathsf{S}^2) : a_*(\pi_1(\mathsf{S}^2))] = [1 : 1] = 1$. □

Problem 2.

No. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and has $\mathsf{d}f(\frac{\partial}{\partial x}) = X, \mathsf{d}f(\frac{\partial}{\partial y}) = Y$. Then

$$x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X = \mathsf{d}f\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial y}, \quad -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = Y = \mathsf{d}f\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\frac{\partial}{\partial y}$$

and this in particular gives the system of equations

$$\frac{\partial f_2}{\partial x} = 1, \quad \frac{\partial f_2}{\partial y} = x.$$

The equation on the left gives $f_2(x, y) = x + g(y)$, for some function $g : \mathbb{R} \to \mathbb{R}$ of y, but the equation on the right gives $f_2(x, y) = xy + h(x)$, for some function $h : \mathbb{R} \to \mathbb{R}$ of x. These two expressions for f_2 can't agree on all of $\mathbb{R}^2$. □

Problem 3.

If there are no points $x \in \mathsf{S}^n$ with $f(x) = x$, then f is free of fixed points, and is thus homotopic to the antipodal map $a : \mathsf{S}^n \to \mathsf{S}^n$ given by $a(x) := -x$, by problem 3 of 2014, Fall. But then $\mathsf{deg}(f) = \mathsf{deg}(a) = (-1)^{n+1}$, a contradiction. Similarly if there are no points $x \in \mathsf{S}^n$ with $f(x) = -x$, then $-f$ is free of fixed points, and is therefore homotopic to a. Then

$$\mathsf{deg}(a)\mathsf{deg}(f) = \mathsf{deg}(a \circ f) = \mathsf{deg}(-f) = \mathsf{deg}(a) \implies \mathsf{deg}(f) = 1,$$

again a contradiction. □

Problem 4 (?)**.**

(a) Let $g : M \times \mathsf{B}^2 \to \mathbb{R}^n$ be given by $(x, y) \mapsto x - f(y)$, and observe that

$$\mathsf{im}(g) = \{v \in \mathbb{R}^n \mid \text{there are } x \in M, y \in \mathsf{B}^2 \text{ so } x = v + f(y)\} = \{v \in \mathbb{R}^n \mid T_v(\mathsf{im}(f)) \cap M \neq \varnothing\}.$$

Now for any $v \in \mathsf{im}(g)$ and any $(x, y) \in g^{-1}(v)$, the map $\mathsf{d}g_{(x,y)}(M \times \mathsf{B}^2) \to \mathsf{T}_v\mathbb{R}^n$ is nonsurjective since $\dim_{\mathbb{R}}(M) \leq n-3$ and $\dim_{\mathbb{R}}(\mathsf{B}^2) = 2$, so $\dim(\mathsf{im}(\mathsf{d}g_{(x,y)})) \leq n-1$. Hence v is a critical value of g. So by Sard, the complement

$$\mathbb{R}^n \setminus \mathsf{im}(g) = \{v \in \mathbb{R}^n \mid T_v(\mathsf{im}(f)) \cap M = \varnothing\}$$

contains all of the regular values of g, and thus has full measure in $\mathbb{R}^n$. Therefore $\mathbb{R}^n \setminus \mathsf{im}(g)$ contains arbitrarily small vectors. □

(b) Take any $g : \mathsf{S}^1 \to \mathbb{R}^n \setminus M$; we're done if we can show that g is nullhomotopic. Consider a (continuous) map $f : \mathsf{B}^2 \to \mathbb{R}^n$ which glues $\partial\mathsf{B}^2$ onto $g(\mathsf{S}^1)$. Since M and $g(\mathsf{S}^1)$ are disjoint compact sets, then there exists an open neighborhood $U \supset g(\mathsf{S}^1)$ disjoint from M.

Thinking of B^2 as the closed unit disc in $\mathbb{C}$, then there's some $\epsilon \in (0, 1/2)$ small enough so that $f^{-1}(U)$ contains the "open collar"

$$C_{2\epsilon} := \{z \in \mathsf{B}^2 \mid \mathsf{dist}(z, \partial\mathsf{B}^2) < 2\epsilon\}.$$

Analogously defining the open collar $C_\epsilon \subset C_{2\epsilon}$, then $\mathsf{B}^2 \setminus C_\epsilon$ is itself homeomorphic to B^2. So by (a) there's an arbitrarily small vector $v \in \mathbb{R}^n$ such that $v + f(\mathsf{B}^2 \setminus C_\epsilon)$ is disjoint from M; since U is open and $f(C_{2\epsilon}) \subset U$, we may choose v small enough so that $v + f(C_{2\epsilon}) \subset U \subset \mathbb{R}^n \setminus M$.

Finally consider the homotopy $\{h_t : \mathsf{B}^2 \to \mathbb{R}^n \setminus M\}_{0 \leq t \leq 1}$ given by

$$h_t(z) := \begin{cases} t\epsilon^{-1}\mathsf{dist}(z, \partial\mathsf{B}^2)v + f(z) & z \in C_\epsilon, \\ tv + f(z) & z \in \mathsf{B}^2 \setminus C_\epsilon. \end{cases}$$

We see that $\{h_t\}_{0 \leq t \leq 1}$ is a homotopy between $h_0 = f$ and the map h_1 which pushes the "inner disc" $f(\mathsf{B}^2 \setminus C_\epsilon)$ by v, fixes the boundary $f(\mathsf{S}^1)$, and continuously connects these images by a collar which lies entirely in $U \subset \mathbb{R}^n \setminus M$.

But $h_1(\mathsf{B}^2)$ is the image of a (contractible) disc, mapped into $\mathbb{R}^n \setminus M$. Hence there exists a further homotopy $\{k_t : \mathsf{B}^2 \to \mathbb{R}^n \setminus M\}_{0 \leq t \leq 1}$ which contracts this image to a point $c \in h_1(\mathsf{B}^2)$, i.e. $k_0 = h_1$ and $k_1 = c$, where $c : \mathsf{B}^2 \to \mathbb{R}^n$ is the constant map to c. The composition of these two homotopies, restricted to the boundary $\partial\mathsf{B}^2$, is a nullhomotopy from g to c. □

Problem 5.

Recall that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$ via the normalization map $u : \mathbb{R}^{n+1} \setminus 0 \twoheadrightarrow \mathsf{S}^n$ given by $u(x) := x/\|x\|$. Hence we have an isomorphism $u^* : \Omega^n(S^n) \to \Omega^n(\mathbb{R}^{n+1} \setminus 0)$, so

$$\int_{\mathsf{S}^n} f^*\omega = \int_{\mathsf{S}^n} f^*u^*(u^*)^{-1}\omega = \int_{\mathsf{S}^n} (u \circ f)^*((u^*)^{-1}\omega) = \mathsf{deg}(u \circ f) \int_{\mathsf{S}^n} (u^*)^{-1}\omega$$

and similarly for g. Therefore as long as the denominator on the left is nonzero,

$$\frac{\int_{\mathsf{S}^n} f^*\omega}{\int_{\mathsf{S}^n} g^*\omega} = \frac{\mathsf{deg}(u \circ f) \int_{\mathsf{S}^n} (u^*)^{-1}\omega}{\mathsf{deg}(u \circ g) \int_{\mathsf{S}^n} (u^*)^{-1}\omega} = \frac{\mathsf{deg}(u \circ f)}{\mathsf{deg}(u \circ g)} \in \mathbb{Q}$$

since $\mathsf{deg}(u \circ f), \mathsf{deg}(u \circ g) \in \mathbb{Z}$. □

Problem 6.

Observing that the solid genus-2 surface W is equivalent to $\bigvee^2 \mathsf{S}^1$,

$$\mathsf{H}_j(W) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 4} & j=1, \\ \mathbb{Z} & j=2, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(S) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 2} & j=1, \\ 0 & \text{else.} \end{cases}$$

By the long exact sequence $\cdots \to \mathsf{H}_j(S) \to \mathsf{H}_j(W) \to \mathsf{H}_j(W,S) \to \mathsf{H}_{j-1}(S) \to \cdots$ for relative homology, we have

$$0 \to \mathsf{H}_3(W,S) \to \mathbb{Z} \to 0 \to \mathsf{H}_2(W,S) \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 4} \xrightarrow{\iota_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\kappa_1} \mathsf{H}_1(W,S) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \xrightarrow{\kappa_0} \mathsf{H}_0(W,S) \to 0$$

and we calculate the relative homologies as follows.

- Immediately, $\mathsf{H}_3(W,S) \cong \mathbb{Z}$.
- $\mathsf{H}_1(S)$ is generated by two lateral loops $[x_1],[x_2]$ and two meridianal loops $[y_1],[y_2]$. The natural inclusion $\iota : S \hookrightarrow W$ maps x_1, x_2 to themselves, so that $\iota_1([x_1]), \iota_1([x_2])$ generate $\mathsf{H}_1(W) \cong \mathbb{Z}^{\oplus 2}$, but includes y_1, y_2 into contractible meridianal discs of W, whereby we have $\iota_1([y_1]) = \iota_1([y_2]) = 0$. Thus $\mathsf{im}(\delta_2) \cong \mathsf{ker}(\iota_1) \cong \mathbb{Z}^{\oplus 2}$, and also $\mathsf{ker}(\delta_2) \cong 0$, so it follows that $\mathsf{H}_2(W,S) \cong \mathbb{Z}^{\oplus 2}$.
- By the above, $\mathsf{ker}(\kappa_1) \cong \mathsf{im}(\iota_1) \cong \mathbb{Z}^{\oplus 2}$, and so $\mathsf{ker}(\delta_1) \cong \mathsf{im}(\kappa_1) \cong 0$. Also ι_0 is injective since it's induced by the inclusion $\iota : W \hookrightarrow S$ of path connected spaces, so $\mathsf{im}(\delta_1) \cong \mathsf{ker}(\iota_0) \cong 0$. Thus $\mathsf{H}_1(W,S) \cong 0$.
- We now have $\mathsf{ker}(\kappa_0) \cong \mathsf{im}(\iota_0) \cong \mathbb{Z}$ since $\mathsf{ker}(\iota_0) \cong 0$. Then $\mathsf{im}(\kappa_0) \cong 0$, and since κ_0 is surjective, then $\mathsf{H}_0(W,S) \cong 0$.

Hence $\mathsf{H}_j(W,S) \cong \begin{cases} 0 & j=0,1, \\ \mathbb{Z}^{\oplus 2} & j=2, \\ \mathbb{Z} & j=3, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 7 (?)**.**

Remark. I think this problem is way too hard for a qualifying exam. Maybe there's an easier approach which didn't occur to me.

Let $n := \mathsf{dim}_{\mathbb{R}}(N)$. We begin with the following construction at some fixed point $x \in M$. Let $V \subset N$ be an open subset containing M.

- Since $M \subset N$ is a codimension-1 submanifold, we can find a connected chart (W, ϕ) centered at x in the maximal atlas of N such that $\phi(W \cap M) = \phi(W) \cap (\mathbb{R}^{n-1} \times 0)$. W.l.o.g., W was chosen small enough so that $W \subset V$. We have an induced homeomorphism

$$\phi_* : \bigwedge^{n-1} \mathsf{T}^*(W \cap M) \to \bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M).$$

Now, $\bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M)$ is a (locally trivial) rank-1 vector bundle and ϕ is a homeomorphism, so we may assume w.l.o.g. that W was also chosen small enough so that we have a trivialization

$$f : \bigwedge^{n-1} \mathsf{T}^*\phi(W \cap M) \to \phi(W \cap M) \times \mathbb{R}.$$

- By our choice of ϕ, we have a natural inclusion $\iota : \phi\big|_{W\cap M}(W \cap M) \hookrightarrow \phi(W) \cap (\mathbb{R}^{n-1} \times 0)$ of codimension 1. And also since $\phi(W)$ is open, we can find a sufficiently small connected neighborhood $\tilde{S} \subset \phi(W \cap M) \times \mathbb{R}$ of $\phi(W \cap M) \times 0$ such that we have an inclusion

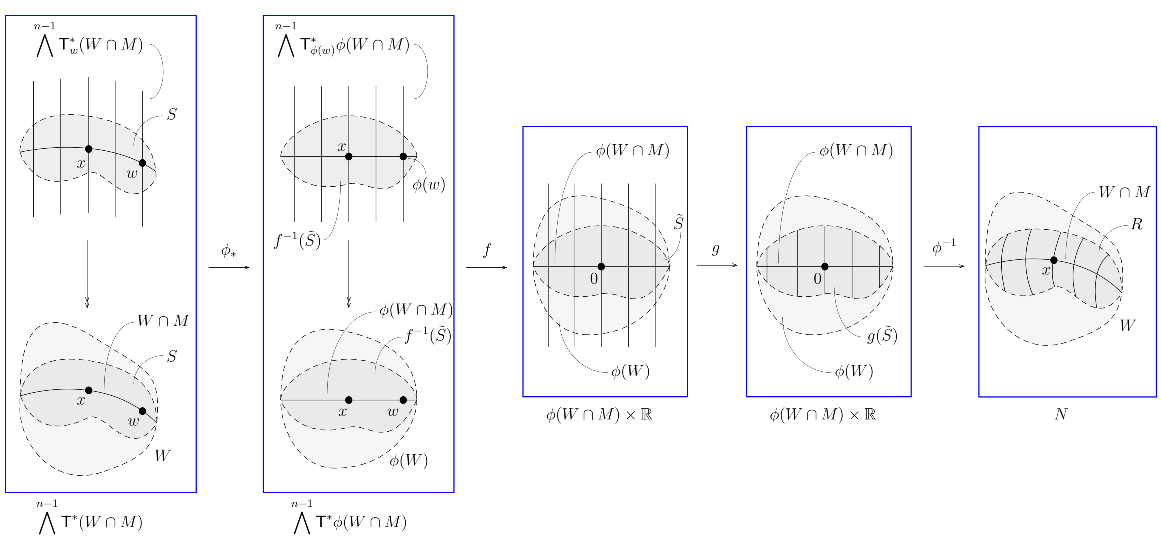
$$g := (\iota \times \mathsf{id}_{\mathbb{R}})\big|_{\tilde{S}} : \tilde{S} \hookrightarrow \phi(W),$$

  which further restricts to a homeomorphism $g : \tilde{S} \to g(\tilde{S})$, denoted again by g.

- Hence the composite $g \circ f \circ \phi_* : \bigwedge^{n-1} \mathsf{T}^*(W \cap M) \to \phi(W)$ restricts to homeomorphisms h and h_0 as in the commutative diagram below. Let S be the preimage $\phi_*^{-1} \circ f^{-1}(\tilde{S})$, let R be the preimage $\phi^{-1} \circ g(\tilde{S})$, and let $0_{W\cap M} : W \cap M \to \bigwedge^{n-1} \mathsf{T}^*(W \cap M)$ be the zero section.

$$\begin{array}{ccccc}
\bigwedge^{n-1} \mathsf{T}^*(W \cap M) & \xrightarrow{g\circ f\circ\phi_*} & \phi(W) & \xrightarrow{\phi^{-1}} & W \\
\uparrow & & \uparrow & & \uparrow \\
S & \xrightarrow{h} & g(\tilde{S}) & \xrightarrow{\phi^{-1}|_{g(\tilde{S})}} & R \\
\uparrow & & \uparrow & & \uparrow \\
S \setminus \mathsf{im}(0_{W\cap M}) & \xrightarrow{h_0} & g(\tilde{S}) \cap (\mathbb{R}^{n-1} \times 0) & \xrightarrow{\phi^{-1}|_{g(\tilde{S})\times(\mathbb{R}^{n-1}\times 0)}} & R \setminus M
\end{array}$$

We now repeat this construction at each point $x \in M$, writing x as a subscript for each of the maps and spaces above to keep track of the base points. Denote also by ϕ_x the restriction $\phi\big|_{R_x}$, for each $x \in M$.



We thus obtain a collection of charts $\{(R_x, \phi_x)\}_{x\in M}$ for M. Clearly $\{R_x\}_{x\in M}$ is an open cover for M, so we may choose a finite subcover $\{R_j\}_{j=1}^m$ by compactness of M, and consider the corresponding collection of charts $\{(R_j, \phi_j)\}_{j=1}^m$. Now $R_1, \ldots, R_m$ are open and connected, and M is also connected, so the union $U := \bigcup_{j=1}^m R_j \supset M$ is itself open and connected.

- Assume that M is orientable. Then $\mathsf{T}^*M \twoheadrightarrow M$ is orientable as a vector bundle, which means that the space $\left(\bigwedge^{n-1} \mathsf{T}^*M\right) \setminus \mathsf{im}(0_M)$ has exactly two connected components. Then

$$S_j \setminus \mathsf{im}(0_{W_j\cap M}) \subset \left(\bigwedge^{n-1} \mathsf{T}^*(W_j \cap M)\right) \setminus \mathsf{im}(0_{W_j\cap M})$$

  also has more than one connected component, for each $1 \le j \le m$. Hence via the composite homeomorphism $S_j \setminus \mathsf{im}(0_{W_j\cap M}) \to R_j \setminus M$ for each $1 \le j \le m$, the space

$$U \setminus M = \left(\bigcup_{j=1}^m R_j\right) \setminus M = \bigcup_{j=1}^m (R_j \setminus M)$$

  is also disconnected.

- Conversely, assume that for every open subset $V' \subset N$ containing M, there's a connected open subset $U' \subset V$ such that $U' \setminus M$ is disconnected. We take V to be the open subset U constructed above, and let $U' \subset U$ be a connected open subset such that $U' \setminus M$ is disconnected. Then upon patching together the disconnected images of the homeomorphisms $R_j \setminus M \to S_j \setminus \mathsf{im}(0_{W_j\cap M})$ for each $1 \le j \le m$, we see that $\left(\bigwedge^{n-1} \mathsf{T}^*M\right) \setminus \mathsf{im}(0_M)$ is disconnected. Choosing a connected component of this space specifies an orientation on M.

□