Fall 2008 $\boxed{\#1}$

$d_f : \Omega^i(M) \to \Omega^{i+1}(M)$, $\quad \omega \mapsto d\omega + df \wedge \omega$, $\quad f$ smooth.

(a) claim: d_f cochain map, i.e. $d_f \circ d_f = 0$.

(b) Let $H_f^i(M)$ be the ith cohomology gp of cochain complex $(\Omega^i(M), d_f)$. Claim: $H_f^0(M) \cong \mathbb{R}$. when M is $\mathbb{R}$.

(a) $$(d_f \circ d_f)(\omega) = d_f(d\omega + df \wedge \omega)$$
$$= d(d\omega + df\wedge\omega) + df \wedge (d\omega + df \wedge \omega)$$
$$= \cancel{dd\omega}^{0} + d(df\wedge\omega) + df\wedge d\omega + \cancel{df\wedge df\wedge\omega}^{0}$$
$$= \cancel{ddf\wedge\omega}^{0} - \cancel{df\wedge d\omega + df\wedge d\omega}^{0}$$
$$= 0.$$

(b) We have

$$0 \xrightarrow{d_f^0} \Omega^0(\mathbb{R}) \xrightarrow{d_f^1} \Omega^1(\mathbb{R})$$

$$H_f^0(\mathbb{R}) = \ker(d_f^1) = \{ g \in \Omega^0(\mathbb{R}) : d_f(g) = 0 \in \Omega^1(\mathbb{R}) \}$$

$$d_f(g) = dg + df \wedge g = dg + g\,df = 0$$

$$dg = -g\,df$$

$H_f^0(\mathbb{R}) = \{$

Fall 2008 $\boxed{\#2}$

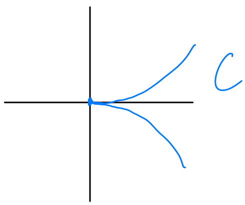
Fall 2008 $\boxed{\#3}$

$C = \{(x,y) : y^2 - x^3 = 0\}$, Claim: C not smooth submfd of $\mathbb{R}^2$.

Hint: space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ tangent to C?

$C = \{(x,y) : y = \pm x^{3/2},\ x \geq 0\}$

$f(x) = x^{3/2} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(0) = 0$



$i : C \hookrightarrow \mathbb{R}^2$

$di_{(x,y)} : T_{(x,y)}C \longrightarrow T_{(x,y)}\mathbb{R}^2$

$\gamma(t) = (t^{2/3}, t) \quad t \in (-\infty, \infty)$

$\gamma'(0) = (\infty, 1)$ undefined.

derivation v at $(0,0)$ defined as map from $C^\infty(\mathbb{R}^2)$ to $\mathbb{R}$ and satisfies product rule. $v(fg) = f(vg) + (vf)g$

Let γ be a curve in C. $\gamma : (-1,1) \to C$

$\gamma'(0) =$

Assume it is a submfd. Then $i : C \hookrightarrow \mathbb{R}^2$ inclusion is an embedding $\Rightarrow$ $di_x : T_x C \to T_{i(x)} \mathbb{R}^2$ injective.

Let $\gamma : I \to C$ with $\gamma(0) = (0,0)$.

$$\gamma(t) = (x(t), y(t)).$$

$$\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) = (x'(t), y'(t))\Big|_{t_0}$$

$$(i \circ \gamma)(t) = ((i \circ x)(t), (i \circ y)(t))$$

$$(i \circ \gamma)'(t) = di_{\gamma(t_0)} d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)$$

Let's assume $x'(t) \neq 0$, then

$$x^3 = y^2 \qquad 3x^2 x' = 2yy'$$

Fall 2008 $\boxed{\#4}$

T = revolving the circle $\{(x,y,z) \mid z=0, (x-R)^2+y^2=r^2\}$ around the y-axis, $R>r$. Compute

$$\int_T x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy.$$

Let ω be the integrand. Then

$$d\omega = dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$$
$$= 3\,dx\wedge dy\wedge dz$$

By Stokes:

$$\int_T \omega = \int_{\partial(D^2\times S^1)} \omega = \int_{D^2\times S^1} d\omega = 3\,\mathrm{vol}(D^2\times S^1)$$

where D^2 is a disk of radius r and S^1 is the circle of radius R.

Then $3\,\mathrm{vol}(D^2\times S^1) = 3\cdot(\pi r^2(2\pi R))$

$$= \boxed{6\pi^2 r^2 R}$$

Fall 2008 $\boxed{\#5}$

B^3 closed 3-d ball. K closed, connected 1-dim'l submfd of B^3 with $\partial K = K \cap \partial B^3 = 2$ points. Compute homology of $B^3 - K$.

Let A be a tubular nbd of K, which exists since it is a 1-dim'l submfd and therefore does not intersect itself. Let $B = B^3 - K$.

Then $A \cap B = A \setminus K$ which deformation retracts to S^1.

And $A \cup B = B^3$ which is contractible.

M.V. gives:

$$\cdots \longrightarrow H_n(A\cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(A \cup B) \longrightarrow \cdots$$

	$H_n(S^1)$	$H_n(\{x_0\}) \oplus H_n(B^3-K)$	$H_n(\{x_0\})$
$n \geq 2$	0	$0 \oplus H_n(B^3-K)$ 0	0
$n = 1$	$\mathbb{Z}$	$0 \oplus H_1(B^3-K)$ $\mathbb{Z}$	0
$n = 0$	$\mathbb{Z}$	$\mathbb{Z} \oplus H_0(B^3-K)$ $\mathbb{Z}$	$\mathbb{Z}$

By exactness $H_2(B^3-K) = 0$. Since $B^3 - K$ is path connected, $H_0(B^3-K) = \mathbb{Z}$. Then we have left

$0 \longrightarrow \mathbb{Z} \longrightarrow H_1(B^3-K) \longrightarrow 0$. Exactness $\Rightarrow$

$H_1(B^3-K) = \mathbb{Z}$. In sum:

$$\boxed{H_n(B^3-K) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{else} \end{cases}}$$

Fall 2008 $\boxed{\#6}$

Recall covering spaces $p: \tilde{X} \to X$, $p': \tilde{X}' \to X$ isomorphic if $\exists$ homeo $\tilde{\phi}: \tilde{X} \to \tilde{X}'$ s.t. $p' \circ \tilde{\phi} = p$. Consider $p: \tilde{X} \to X$ of torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?

"We know that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X; x_0) \to \Sigma_n$, the symmetric group on n elements, where the equivalence relation identifies a homomorphism ρ with each of its conjugates $h^{-1}\rho h$ by elements $h \in \Sigma_n$." - Hatcher

Consider $\rho: \mathbb{Z}\langle a\rangle \times \mathbb{Z}\langle b\rangle \to \Sigma_3$ homomorphism, where $\Sigma_3 = \{(), (12), (13), (23), (123), (132)\}$. ρ is determined by $\rho(a), \rho(b)$.

Conjugates: $\overset{C_1}{\{(12), (13), (23)\}}$ and $\overset{C_2}{\{(123), (132)\}}$

- $\rho(a) = \rho(b) = ()$
- $\rho(a) = ()$, $\rho(b) \in C_1$
- $\rho(a) = ()$, $\rho(b) \in C_2$
- $\rho(a) \in C_1$, $\rho(b) = ()$
- $\rho(a) = \rho(b) \in C_1$
- $\rho(a) = \rho(b) \in C_2$
- $\rho(a) \neq \rho(b) \in C_2$
- $\rho(a) \neq \rho(b) \in C_1$ 2

• $\rho(a) \in C_2$, $\rho(b) = ()$

• $\rho(a) \in C_1$ $\rho(b) \in C_2$

• $\rho(a) \in C_2$ $\rho(b) \in C_1$

$$\begin{array}{ccccc}
((12),(123)) & \sim & ((23),(132)) & \sim & ((13),(123)) \\
\wr & & \wr & & \wr \\
((12),(132)) & \sim & ((23),(123)) & \sim & ((13),(132))
\end{array}$$

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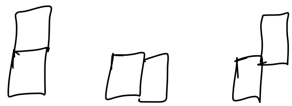
3 2-sheeted coverings

3 (2+1)-sheeted coverings



1 (1+1+1)-sheeted covering