1. a) $d \circ d = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\text{ord}\,\alpha} \alpha \wedge d\beta$

   b) $\ker d_p = \{ce^{f(x)} \in \Omega^0(\mathbb{R})\} \cong \mathbb{R}$.

2. Pull volume forms on S^n, S^m back to S^{n+m}. Use de Rham of S^{m+n} to conclude they must be exact.

3. Analysis: $\gamma_1(t) = v_1 t + \hat{\gamma}_1(t)$ etc...

4. Stokes

5. ~~$\mathbb{R}^3 \setminus K \overset{\text{h.e.}}{\simeq} S^1$~~ Inverse Mayer Vietoris and Tubular neighborhoods

6. Classification of n-sheeted covering spaces.

1. Consider $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ where M smooth mfld,
$\omega \longmapsto d\omega + df \wedge \omega$ f smooth fn.

(a) Show d_f is a cochain map, i.e. $d_f \circ d_f = 0$
(b) Let $H^i_f(M)$ be the i^{th} cohomology group of the cochain complex (Ω^i, d_f). Show $H^0_f(M) \cong \mathbb{R}$ when $M = \mathbb{R}$..!

(a) Recall that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$.

$$\begin{aligned}
d_f \circ d_f(\omega) &= d_f(d\omega + df \wedge \omega) \\
&= d(d\omega + df \wedge \omega) + df \wedge (d\omega + df \wedge \omega) \\
&= \cancelto{0}{d(d\omega)} + d(df \wedge \omega) + df \wedge (d\omega + df \wedge \omega) \\
&= \cancelto{0}{d(df)} \wedge \omega + (-1)^{\deg(df)} df \wedge d\omega + df \wedge (d\omega + df \wedge \omega) \\
&= \cancelto{0}{-df \wedge d\omega + df \wedge d\omega} + df \wedge df \wedge \omega \\
&= df \wedge df \wedge \omega = 0 \quad \text{since } df \wedge df = -df \wedge df \\
&\qquad\qquad\qquad \Rightarrow df \wedge df = 0
\end{aligned}$$

(b) Consider the cochain complex: (df 1-form)

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d_f} \Omega^1(\mathbb{R}) \implies H^0_f(\mathbb{R}) \cong \ker d_f.$$

So consider this kernel: $\ker d_f = \{ g \in \Omega^0(\mathbb{R}) : d_f g = 0 \}$

$$\Rightarrow 0 = d_f(g) = dg + df \wedge g \Rightarrow dg = -df \wedge g = g \wedge df = g\,df$$

$$\Rightarrow g = Ae^f,$$

hence $\ker d_f = \{Ae^{f(x)}\}_{A \in \mathbb{R}} \cong \mathbb{R}$.

$$\Rightarrow H^0_f(\mathbb{R}) \cong \mathbb{R}$$

(2.) Show that the homomorphism $f^*: H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$

induced in de Rham cohomology by $f: S^{m+n} \to S^m \times S^n$

is trivial for all $k>0$, $m,n>0$.

First, note that $H^k(S^{m+n}) \equiv 0$ for all $k < m+n$, hence we only need to be concerned with the map:

$$f^*: H^{m+n}(S^m \times S^n) \longrightarrow H^{m+n}(S^{m+n})$$

Recall that both S^n and S^m have volume forms: $\begin{cases} \omega_m \in \Omega^m(S^m) \\ \omega_n \in \Omega^n(S^n) \end{cases}$,

which we know generate $H^m(S^m)$, $H^n(S^n)$ (resp.) by the top-dim'l de Rham theorem. Now consider their pullbacks under the respective projection maps $\pi_n: S^n \times S^m \to S^n$, $\pi_m: S^n \times S^m \to S^m$:

$$\pi_n^*: H^n(S^n) \longrightarrow H^n(S^n \times S^m), \quad \omega_n \longmapsto \pi_n^*(\omega_n)$$

$$\pi_m^*: H^m(S^m) \longrightarrow H^m(S^n \times S^m), \quad \omega_m \longmapsto \pi_m^*(\omega_m)$$

and take the wedge product of the two:

$$\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m) \in H^{n+m}(S^n \times S^m)$$

See now that ω_n is of the form $\omega_n = \underbrace{f\,dx_1 \wedge \cdots \wedge dx_n}_{\text{for coord. } (x_i) \text{ on } S^n}$, $f \neq 0$,

and so
$$\pi_n^*(f\,dx_1 \wedge \cdots \wedge dx_n) = \pi_n^*(f)\,\pi_n^*(dx_1) \wedge \cdots \wedge \pi_n^*(dx_n)$$

(?)
$$= (f \circ \pi_n)\,dx_1 \wedge \cdots \wedge dx_n$$

and similarly for ω_m and $\pi_m^*(\omega_m)$, hence their wedge product is of the form $F\,dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_m$ with $F \neq 0$, hence $\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m)$ is a volume form on $\underbrace{S^n \times S^m}_{\text{coords } (x_i, y_j)}$.

for coord. (y_i) on S^m

2 cont'd:

So we have a volume form $\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m) \in H^{n+m}(S^n \times S^m)$, i.e. a generator of $H^{n+m}(S^n \times S^m)$.

Now see that:

$$f^*(\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m)) = f^*\pi_n^*(\omega_n) \wedge f^*\pi_m^*(\omega_m).$$
$$= 0 \wedge 0 = 0$$

Since $f^*: H^n(S^n \times S^m) \to H^n(S^{n+m}) \cong 0$
$f^*: H^m(S^n \times S^m) \to H^m(S^{n+m}) \cong 0$,

hence $f^*: H^{n+m}(S^n \times S^m) \to H^{n+m}(S^{n+m})$ is the zero map, as was desired.

③ Show that $C = \{(x,y) : y^2 = x^3\}$ is not a smooth submfld of $\mathbb{R}^2$:

Consider a curve $\alpha : (-\varepsilon, \varepsilon) \to C$ with $\alpha(0) = (0,0)$.

Then $\alpha(t) = (x(t), y(t))$; hence $x(t)^3 = y(t)^2$.

These constituent functions are smooth, hence representable as Taylor series:

Suppose $y(t) = ct + O(t^2)$ and $x(t) = O(t)$ (since $x(0) = 0 = y(0)$)

Then: $x(t)^3 = y(t)^2 \Rightarrow O(t^3) = (ct + O(t^2))^2$

$$= c^2t^2 + O(t^3)$$

$$\Rightarrow O(t^3) = c^2t^2 \Rightarrow c^2 = 0 \Rightarrow c = 0$$

Hence $y(t) = O(t^2) \Rightarrow y'(t) = O(t)$

$$\Rightarrow \underline{y'(0) = 0}$$

So we've shown $y(t) = O(t^2)$.

Now suppose $x(t) = dt + O(t^2)$. Then:

$$(dt + O(t^2))^3 = (O(t^2))^2$$

$$\Rightarrow d^3t^3 + O(t^4) = O(t^4) \Rightarrow d^3t^3 = O(t^4)$$

$$\Rightarrow d^3 = 0 \Rightarrow d = 0$$

So then $x(t) = O(t^2) \Rightarrow x'(t) = O(t) \Rightarrow \underline{x'(0) = 0}$

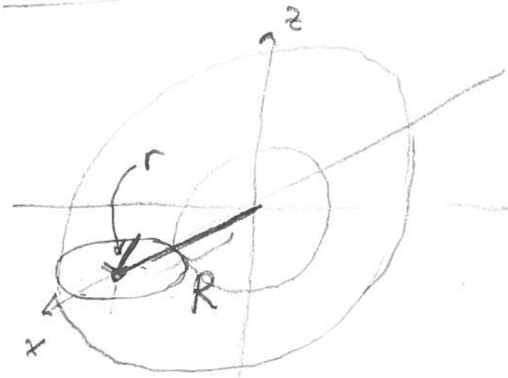
Hence we've shown the tangent vector to any smooth curve thru $(0,0)$ in C is $(0,0)$, i.e.

$T_{(0,0)}C = 0 \Rightarrow \dim T_{(0,0)}C = 0$, a contradiction

since if C submfld, $\dim T_{(0,0)} = 1$

(4.) T = revolve $\{(x,y,z): z=0, (x-R)^2+y^2=r^2\}$ around y-axis, and $R>r$. Compute:

$$\int_T x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy .$$



T is a torus.

Let K = solid torus s.t. $\partial K = T$.

Now see that $d(x\,dy\wedge dz - y\,dx\wedge dz + x\,dx\wedge dy)$

$$= dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$$
$$= dx\wedge dy\wedge dz + dx\wedge dy\wedge dz + dx\wedge dy\wedge dz$$
$$= 3\,dx\wedge dy\wedge dz$$

and apply Stokes:

$$\int_T (x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) = \int_{\partial K} (x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$$

$$= \int_K d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) = \int_K 3\,dx\wedge dy\wedge dz$$

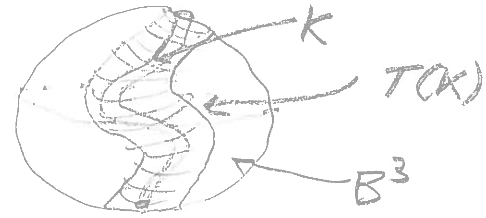
$$= 3\int_K dx\wedge dy\wedge dz = 3\,\mathrm{Vol}(K) = (\pi r^2)(2\pi R) = \underline{2\pi^2 R r^2}$$

(5) B^3 closed ball, $K \subseteq B^3$ closed, connected, 1-dim submfld such that $\partial K = K \cap \partial B^3 = 2$ pts (apple minus wormhole)

Compute $H_i(B^3 \setminus K)$:

Let A = tubular nbhd of $K := T(K)$

$B = B^3 \setminus K$



Now note that K is a submanifold, hence it will not have any self-intersections since those would not be locally Euclidean (i.e. $\nexists$ homeo.

)

Therefore K can be contracted to a point (as a submfld of $\mathbb{R}^3$), hence $T(K) \simeq pt$.

Now, by construction, $K \subseteq T(K)$ such that $T(K) \setminus K \simeq$ cylinder $\simeq S^1$, hence

$$A \cap B \simeq T(K) \setminus K \simeq S^1$$

Finally, clearly $A \cup B = B^3 \simeq pt$.

So we have:

$$H_i(A) \cong H_i(pt) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases} \qquad H_i(A \cap B) \cong H_i(S^1) \cong \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$$

$$H_i(B) \cong H_i(B^3 \setminus K) \qquad H_i(A \cup B) \cong H_i(pt) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$$

Now apply Mayer-Vietoris:

$$0 \to H_2(A\cap B) \to H_2(A)\oplus H_2(B) \to H_2(A\cup B) \to H_1(A\cap B)$$
$$\to H_1(A)\oplus H_1(B) \to H_1(A\cup B) \to H_0(A\cap B) \to H_0(A)\oplus H_0(B)$$
$$\to H_0(A\cup B) \to 0$$

$$\Rightarrow\ 0 \to 0 \to 0\oplus H_2(B^3\setminus K) \to 0 \to \mathbb{Z} \to 0\oplus H_1(B^3\setminus K)$$
$$\to 0 \to \mathbb{Z} \to \mathbb{Z}\oplus H_0(B^3\setminus K) \to \mathbb{Z} \to 0$$

$\Rightarrow$ $H_2(B^3\setminus K)\cong 0$, $H_1(B^3\setminus K)\cong \mathbb{Z}$ by exactness

$H_0(B^3\setminus K)\cong\mathbb{Z}$ by path-connectedness.

(6.) $p:\tilde{X}\to X$, $p':\tilde{X}'\to X$ isomorphic if $\exists$ homeo. $\phi:\tilde{X}\xrightarrow{\sim}\tilde{X}'$ s.t. $p'\circ\phi=p$.

Consider the covering space $p:\tilde{X}\to T^2$ of the torus whose fibers $p^{-1}(x_0)$ at any pt. $x_0\in T^2$ consists of 3 points.

How many isomorphism classes of such a covering?

Recall that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X)\to S_n$.

Here, $X=T^2$ and $n=3$, hence consider homomorphisms

$$\pi_1(T^2)\to S_3,\ \text{i.e. } \mathbb{Z}\oplus\mathbb{Z}\to S_3.$$

See that $\pi_1(T^2)\cong\mathbb{Z}\oplus\mathbb{Z}\cong\mathbb{Z}\{a\}\oplus\mathbb{Z}\{b\}$, i.e. has two generators a,b; the group is free abelian, hence the generators commute: $ab=ba$, hence for any hom. $f:\mathbb{Z}\oplus\mathbb{Z}\to S_3$, $f(a)f(b)=f(ab)=f(ba)=f(b)f(a)$, hence f maps the generators to commuting pairs $\sigma,\tau\in S_3$.

- Case $f(a)=f(b)$: Clearly two equal permutations commute; there are 6 elts in S_3, so this case represents 6 iso. classes.
- Commuting pairs: the following are the commuting pairs in S_3:

$$\left\{\begin{array}{l}(id,(12))\\(id,(13))\\(id,(23))\\(id,(123))\\(id,(132))\end{array}\right.\ \text{are clear.}$$

- Also see that $(123)=(132)^2$, hence $(123)(132)=(132)^2(132)$
$=id=(132)(132)^2$
$=(132)(123)$,

so we also have
$((123),(132))$.

For a total of 6 commuting pairs, and 12 total cases since pairs may $=(f(a),f(b))$ or $=(f(b),f(a))$. Thus $12+6=18$ iso. classes