2008, Fall

Problem 1.

Background. Certain references may call $(\Omega^\bullet(M), \mathsf{d}_f^\bullet)$ the *f-twisted de Rham cochain complex of M*. In this problem we show that d_f is indeed a coboundary operator, and compute the 0th cohomology of this complex for $\mathbb{R}$.

(a) Observe that $\mathsf{d}f \wedge \mathsf{d}f = \mathsf{d}(f \wedge \mathsf{d}f) = -\mathsf{d}(\mathsf{d}f \wedge f) = -\mathsf{d}f \wedge \mathsf{d}f$, and so $\mathsf{d}f \wedge \mathsf{d}f = 0$. Then

$$\begin{aligned}\mathsf{d}_f^2\omega &= \mathsf{d}_f(\mathsf{d}\omega + \mathsf{d}f \wedge \omega) = \mathsf{d}(\mathsf{d}\omega + \mathsf{d}f \wedge \omega) + \mathsf{d}f \wedge (\mathsf{d}\omega + \mathsf{d}f \wedge \omega) \\ &= \underbrace{\mathsf{d}^2\omega}_{=0} + \underbrace{\mathsf{d}^2 f}_{=0} \wedge\omega - \mathsf{d}f \wedge \mathsf{d}\omega + \mathsf{d}f \wedge \mathsf{d}\omega + \underbrace{\mathsf{d}f \wedge \mathsf{d}f}_{=0} \wedge\omega = 0\end{aligned}$$

for any $\omega \in \Omega^j(M)$. ☐

(b) Suppose $g \in \ker((\mathsf{d}_f)_0) \subset \Omega^0(M) \cong \mathsf{C}^\infty(\mathbb{R})$. Then $0 = \mathsf{d}_f g = \mathsf{d}g + \mathsf{d}f \wedge \mathsf{d}g = \mathsf{d}g - g\mathsf{d}f$, so $g = \mathsf{d}g/\mathsf{d}f$ and hence $g = c_g e^f$ for some constant $c_g \in \mathbb{R}$. Conversely, any $g \in \mathsf{C}^\infty(\mathbb{R})$ of this form clearly satisfies $\mathsf{d}_f g = 0$. Hence the assignment $g \mapsto c_g$ is a one-to-one correspondence from $\mathsf{H}^0_f(\mathbb{R}) \cong \ker((\mathsf{d}_f)_0)$ to $\mathbb{R}$, which completes the argument. ☐

Problem 2.

We need only check that the map $f^* : \mathsf{H}^{m+n}_{\mathsf{dR}}(\mathsf{S}^m \times \mathsf{S}^n) \to \mathsf{H}^{m+n}_{\mathsf{dR}}(\mathsf{S}^{m+n})$ is trivial, since we know that $\mathsf{H}^j_{\mathsf{dR}}(\mathsf{S}^{m+n}) \cong 0$ for all $j \geq 1$ with $j \neq m+n$. Given volume forms $\alpha \in \Omega^m(\mathsf{S}^m)$ and $\beta \in \Omega^n(\mathsf{S}^n)$, the canonical projections $\pi_m : \mathsf{S}^m \times \mathsf{S}^n \twoheadrightarrow \mathsf{S}^m$ and $\pi_n : \mathsf{S}^m \times \mathsf{S}^n \twoheadrightarrow \mathsf{S}^n$ yield the two nonzero forms $\pi_m^*\alpha \in \Omega^m(\mathsf{S}^m \times \mathsf{S}^n)$ and $\pi_n^*\beta \in \Omega^n(\mathsf{S}^m \times \mathsf{S}^n)$. It's easy to verify that $(\pi_m^*\alpha) \wedge (\pi_n^*\beta)$ is a volume form on $\mathsf{S}^m \times \mathsf{S}^n$, whereby $[(\pi_m^*\alpha) \wedge (\pi_n^*\beta)]$ generates $\mathsf{H}^{m+n}_{\mathsf{dR}}(\mathsf{S}^m \times \mathsf{S}^n)$. Then f^* is trivial if it maps this generator to 0. To see this, recall that f^* is trivial on $\mathsf{H}^m_{\mathsf{dR}}(\mathsf{S}^m \times \mathsf{S}^n)$ and $\mathsf{H}^n_{\mathsf{dR}}(\mathsf{S}^m \times \mathsf{S}^n)$, whereby

$$f^*[(\pi_m^*\alpha) \wedge (\pi_n^*\beta)] = \underbrace{(f^*[\pi_m^*\alpha])}_{=0} \wedge \underbrace{(f^*[\pi_n^*\beta])}_{=0} = 0$$

as desired. ☐

Problem 3 (?)**.**

Remark. It may be tempting to try to exhibit C as the preimage of 0 under $f(x,y) := y^2 - x^3$ and observe that $\mathsf{d}f_{(0,0)}$ is nonsurjective. However, this wouldn't prove that C isn't a submanifold of $\mathbb{R}^2$, but only that f was the wrong choice of function; *a priori* we may have that $C = g^{-1}(p)$ for some other smooth function g and some regular value p of g.

Assume that C is a submanifold of $\mathbb{R}^2$. Then by the implicit function theorem, on a sufficiently small neighborhood of the point $(0,0) \in C$, we can write y as a function of x. By definition of C, this function must be $y = \pm x^{3/2}$. But on any neighborhood of 0 on the x-axis, this isn't a function since it assigns two values to any $x > 0$. ☐

Problem 4.

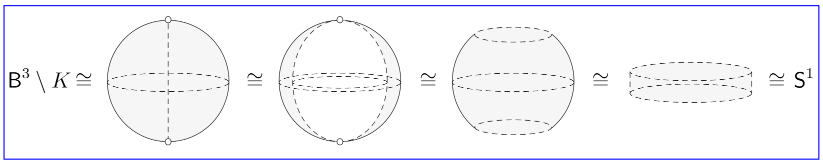
Let $X \subset \mathbb{R}^3$ be the solid torus with $\partial X = T$, and let $\omega := x\mathsf{d}y \wedge \mathsf{d}z - y\mathsf{d}x \wedge \mathsf{d}z + z\mathsf{d}x \wedge \mathsf{d}y$. Then $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$, and

$$\int_T \omega = 3\int_X \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(X) = 3(2\pi R)(\pi r^2) = 6\pi^2 r^2 R$$

by Stokes. ☐

Problem 5.

We first stretch out the missing curve K inside B^3 until we hollow out the inside of the sphere, leaving us a copy of S^2 with two points removed. We then stretch out each of these missing points along the surface and toward the equator; the result is equivalent to a circle as shown.



Thus $\mathsf{H}_j(\mathsf{B}^3 \setminus K) \cong \mathsf{H}_j(\mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$ □

Problem 6.

The 3-sheeted covers of X are classified by equivalence classes of homomorphisms $\pi_1(X) \to \Sigma_3$. Note that if f is such a homomorphism, then f is completely determined by where it sends the two generators x, y of $\pi_1(X) \cong \pi_1(\mathsf{S}^1 \times \mathsf{S}^1) \cong \mathbb{Z}^{\oplus 2}$; and, $f(x)f(y) = f(xy) = f(yx) = f(y)f(x)$ since $\mathbb{Z}^{\oplus 2}$ is abelian. So these homomorphisms are in bijection with ordered pairs of elements $(\alpha, \beta) \in \Sigma_3$ with $\alpha\beta = \beta\alpha$, and we turn our attention to counting these pairs.

- Any of the six elements of Σ_3 commutes with itself, so this gives us six ordered pairs of the form (α, α), with $\alpha \in \Sigma_3$.
- Any of the six elements of Σ_3 commutes with $1 \in \Sigma_3$, so this gives us five new unordered pairs of the form $\{1, \alpha\}$, with $\alpha \in \Sigma_3$, and hence ten new ordered pairs. (We already counted the pair $(1, 1)$ in the previous step.)
- Finally, it is routinely verified that the two 3-cycles in Σ_3 commute, so we have two new ordered pairs $((123), (132))$ and $((132), (123))$.

In all, we've counted 18 pairs of the desired form, and from this we conclude that there are exactly 18 3-sheeted covers of X. □