Fall 2007 $\boxed{\#1}$

X path connected. $H_p(X;\mathbb{Z})=0 \quad \forall \quad 0<p\leq n$.

Compute $H_p(X\times S^n;\mathbb{Z}) \quad \forall \quad 0<p\leq n$.

Let $A = X\times(S^n\setminus N)$ and $B = X\times(S^n\setminus S)$ for N, S the north and south poles of S^n.

Then $A\cap B = X\times(S^n\setminus\{N,S\})$ def. retracts to $X\times S^{n-1}$.

A, B def retract to $X\times\{S\}$, $X\times\{N\}$ both $\simeq X$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_p(A\cap B) & \longrightarrow & H_p(A)\oplus H_p(B) & \longrightarrow & H_p(X\times S^n) & \longrightarrow & \cdots \\
& \wr\wr & & \wr\wr & & \wr\wr & & \\
\cdots \longrightarrow & H_p(X\times S^{n-1}) & \longrightarrow & H_p(X)\oplus H_p(X) & \longrightarrow & H_p(X\times S^n) & \longrightarrow & \cdots
\end{array}$$

So we get for $1<p\leq n$

$$\underbrace{H_p(X)\oplus H_p(X)}_{0} \longrightarrow H_p(X\times S^n) \longrightarrow H_{p-1}(X\times S^{n-1}) \longrightarrow \underbrace{H_{p-1}(X)\oplus H_{p-1}(X)}_{0}$$

implying $H_p(X\times S^n) \approx H_{p-1}(X\times S^{n-1}) \quad \forall \quad 1<p\leq n$.

Thus $H_p(X\times S^n) \approx H_1(X\times S^{n-p+1}) \quad \forall \quad 1<p\leq n$.

We have

$$\underbrace{H_1(X)\oplus H_1(X)}_{0} \xrightarrow{i_*} H_1(X\times S^{n-p+1}) \xrightarrow{\partial} \underbrace{H_0(X\times S^{n-p})}_{\begin{cases}\mathbb{Z} & 0<p<n \\ \mathbb{Z}^2 & p=n\end{cases}} \xrightarrow{i_*} \underbrace{H_0(X)\oplus H_0(X)}_{\mathbb{Z}^2}$$

$$\begin{cases} a \longmapsto (a,a) \\ (a,b) \longmapsto (a+b, a+b) \end{cases}$$

$\ker \partial = \operatorname{im} j_* = 0 \Rightarrow \partial$ injective.

$$\operatorname{im} \partial = \ker i_* = \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

$$\Rightarrow H_1(X \times S^{n-p+1}) = \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

Hence, $$H_p(X \times S^n) \approx \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

Fall 2007 $\boxed{\#2}$

C_0, C_1 disjoint circles in $\mathbb{R}^3$. $A = S^1 \times [0,1]$ cylinder.

$X = \mathbb{R}^3 \sqcup A$ gluing $S^1 \times \{0\}$ to C_0 and $S^1 \times \{1\}$ to C_1 by homeomorphisms f_0, f_1.

Compute $\pi_1(X)$.

Let $U = \mathbb{R}^3$, $V = C_0' \cup C_1' \cup A \cup \gamma'$ for γ a path connecting C_0 and C_1, and C_0', C_1', γ' small enough neighborhoods to maintain distinctness of C_0', C_1' and keep V open. Then $U, V, U \cap V$ path connected, U, V open, and $U \cup V = X$. Hence SVK gives

$\pi_1(X) = \pi_1(\mathbb{R}^3) * \pi_1(C_0' \cup C_1' \cup A \cup \gamma') / \langle i(\omega) j(\omega)^{-1} \rangle$

for $\omega \in \pi_1(U \cap V)$, $i: \pi_1(U \cap V) \hookrightarrow \pi_1(U)$, $j: \pi_1(U \cap V) \hookrightarrow \pi_1(V)$ the induced inclusion maps.

$\pi_1(\mathbb{R}^3) = 1$. $C_0' \cup C_1' \cup A \cup \gamma' \simeq S^1 \vee S^1$ so $\pi_1(V) = \mathbb{Z} * \mathbb{Z}$. and $U \cap V \simeq S^1 \vee S^1$ so $\pi_1(U \cap V) = \mathbb{Z} * \mathbb{Z}$.

Let $a \in \pi_1(U \cap V)$ be a loop around C_0 and let $b \in \pi_1(U \cap V)$ be the loop $\gamma \cdot C_1 \cdot \gamma^{-1}$. Then in $\mathbb{R}^3$ these loops are contractible, but in V, a is sent to one of the generators c of $\pi_1(V)$. b is sent to dcd^{-1} for generators c, d of $\pi_1(V)$. Hence

$\pi_1(X) = \mathbb{Z} * \mathbb{Z} / \langle c, (dcd^{-1})^{-1} \rangle = \langle c, d \mid c = dcd^{-1} = 1 \rangle = \boxed{\mathbb{Z}}$.

Fall 2007 $\boxed{\#3}$

$M_n(\mathbb{R})$ vector space of $n \times n$ matrizes w/ coeffs in $\mathbb{R}$.

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$. Compute $d(\det)_{I_n}$ for $I_n \in M_n(\mathbb{R})$ identity.

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \longleftrightarrow (x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}) \in \mathbb{R}^{n^2}$$

$$d(\det)_{I_n} = \left[\frac{\partial(\det)}{\partial x_{11}}\bigg|_{I_n} \quad \cdots \quad \frac{\partial(\det)}{\partial x_{nn}}\bigg|_{I_n} \right]$$

$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} \det(A_{\hat{ij}})$ for any $i \in \{1, \ldots, n\}$,

where $A_{\hat{ij}}$ is the matrix minor leaving out the ith row and jth column. $\det(I_{n\hat{ij}}) = 0$ unless $i = j$ since $I_{n\hat{ij}}$ has a row and column of 0's if $i \neq j$.

And $\det(I_{n\hat{ii}}) = 1 \quad \forall\, i \in \{1, \ldots, n\}$.

Thus $$\frac{\partial(\det)}{\partial x_{ij}}\bigg|_{I_n} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Thus for any $A \in T_{I_n} M_n(\mathbb{R})$, with $A = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$

$d(\det)_{I_n}(A) = \sum_{i=1}^{n} x_{ii} = tr(A)$.

Hence $d(\det)_{I_n} : T_{I_n} M_n(\mathbb{R}) \longrightarrow T_1 \mathbb{R}$ is the trace map.

$$\begin{array}{ccc} T_{I_n} M_n(\mathbb{R}) & \longrightarrow & T_1 \mathbb{R} \\ \wr\wr & & \wr\wr \\ M_n(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ A & \longmapsto & tr(A). \end{array}$$

Fall 2007 $\boxed{\# 4}$

M compact, orientable n-dim'l mfd

$f : \partial M \xrightarrow{\cong} S^{n-1} \subset \mathbb{R}^n$. $F : M \to \mathbb{R}^n$ continuous

with $F|_{\partial M} = f$. Claim: $F(M)$ contains the center O of S^{n-1}.

Assume not. Then $F(M)$ deformation retracts onto S^{n-1}. Call this retraction r. Then, we have

$$S^{n-1} \xrightarrow{f^{-1}} \partial M \xhookrightarrow{i} M \xrightarrow{F} \mathbb{R}^n - \{0\} \xrightarrow{r} S^{n-1}$$

where i is the inclusion. This composition is the identity map since $r(F(i(f^{-1}(x)))) = r(F(f^{-1}(x))) = r(x) = x$.

Then the induced homomorphism $(r \circ F \circ i \circ f^{-1})_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity map. But $H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M)$ is the 0-map, so this is a contradiction.

Fall 2007 $\boxed{\#5}$

$\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 10\}$ $\omega = \dfrac{x\,dy - y\,dx}{4x^2+y^2}$

a) Show ω closed in Ω.

b) Show ω not exact in Ω.

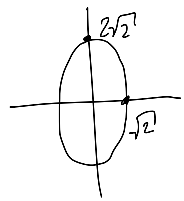
---

a) $$d\omega = d\left(\frac{x}{4x^2+y^2}\right)\wedge dy - d\left(\frac{y}{4x^2+y^2}\right)\wedge dx$$

$$= \frac{4x^2+y^2-8x^2}{(4x^2+y^2)^2}\,dx\wedge dy - \frac{4x^2+y^2-2y^2}{(4x^2+y^2)^2}\,dy\wedge dx$$

$$= \frac{-4x^2+y^2+4x^2-y^2}{(4x^2+y^2)^2}\,dx\wedge dy = 0 \implies \omega \text{ closed in } \Omega.$$

b) Consider ellipse $E: 4x^2+y^2 = 8 \subset \Omega$ which is parametrized by



$$(x,y) = (\sqrt{2}\cos\theta,\ 2\sqrt{2}\sin\theta)$$

Then $dx = -\sqrt{2}\sin\theta\,d\theta$ and $dy = 2\sqrt{2}\cos\theta\,d\theta$

So $$\omega = \frac{4\cos^2\theta\,d\theta + 4\sin^2\theta\,d\theta}{8} = \frac{1}{2}\,d\theta$$

Thus, $$\int_E \omega = \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi \neq 0.$$

If ω exact in E, then $\omega = d\alpha$ and Stokes gives

$$\int_E \omega = \int_E d\alpha = \int_{\partial E}\alpha = \int_{\emptyset}\alpha = 0,$$ a contradiction.

So ω not exact in E. And since $E \subset \Omega$,

$\implies \omega$ not exact in Ω.

Fall 2007 $\boxed{\#6}$

$\varphi : \mathbb{R}^2 \to \mathbb{RP}^2 \quad (x,y) \longmapsto$ line passing through $(x,y,1)$.

$C = \{(x,y) \in \mathbb{R}^2 ;\ y^2 = x^3 - x\}$.

Claim: $\overline{\varphi(C)}$ diff. submfd of $\mathbb{RP}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ send $(x,y) \longmapsto -x^3 + x + y^2$

Then $C = f^{-1}(0)$. Is 0 a regular value of f?

$$df_p = \left[\frac{\partial f}{\partial x}\Big|_p \ \ \frac{\partial f}{\partial y}\Big|_p \right] = \left[-3x^2+1 \ \ 2y \right]_p \quad \text{for } p \in C.$$

$2y = 0 \Rightarrow y = 0 \Rightarrow -x^3 + x = 0 \Rightarrow x \in \{-1, 0, 1\}$

$\Rightarrow -3x^2 + 1 \in \{-2, 1\} \Rightarrow df_p \neq 0 \ \ \forall\, p \in C$.

Hence C is a submfd of $\mathbb{R}^2$ by regular val. Thm.

We see φ is a diffeomorphism onto its image $\varphi(\mathbb{R}^2)$ and $\overline{\varphi(\mathbb{R}^2)} = \mathbb{RP}^2$. Hence $\varphi(C)$ is a submanifold of $\varphi(\mathbb{R}^2)$ and $\overline{\varphi(C)}$ is a submanifold of $\overline{\varphi(\mathbb{R}^2)} = \mathbb{RP}^2$.

? Does the last part work?

Attempt #2

$\varphi : \mathbb{R}^2 \to \mathbb{RP}^2 \quad (x,y) \longmapsto$ line passing through $(x,y,1)$.

$C = \{(x,y) \in \mathbb{R}^2 ; \ y^2 = x^3 - x\}$.

Claim: $\overline{\varphi(C)}$ diff. submfd of $\mathbb{RP}^2$.

$\frac{y^2}{x^2} = x - \frac{1}{x}$

$\frac{y}{x} = \sqrt{x - \frac{1}{x}} \to \pm\infty$ as $x \to \infty$.

$$\varphi(C) = \{ [x,y,1] \in \mathbb{RP}^2 : -x^3 + x + y^2 = 0 \}$$

$$= \{ [x,y,z] \in \mathbb{RP}^2 : -\left(\tfrac{x}{z}\right)^3 + \tfrac{x}{z} + \left(\tfrac{y}{z}\right)^2 = 0, \ z \neq 0 \}$$

$$= \{ [x,y,z] \in \mathbb{RP}^2 : -x^3 + z^2 x + z y^2 = 0, \ z \neq 0 \}$$

Consider $\{ [x,y,z] \in \mathbb{RP}^2 : -x^3 + z^2 x + z y^2 = 0 \}$.

Then $z = 0 \Rightarrow x = 0 \Rightarrow [0,1,0]$ is the only element of $\mathbb{RP}^2$ with $z=0$ in this set. Hence this set $= \varphi(C) \cup \{[0,1,0]\}$. And it is $f^{-1}(0)$ for $f : \mathbb{RP}^2 \to \mathbb{R}$ defined by $f([x,y,z]) = -x^3 + z^2 x + z y^2$, which is well defined since each term has degree 3, $\varphi(C) \cup \{[0,1,0]\}$ is closed since it is the preimage of a closed set $\{0\} \subset \mathbb{R}$. And since it is the smallest closed set containing $\varphi(C)$, it is $\overline{\varphi(C)}$.

Is 0 a regular value of f?

$df_p = \begin{bmatrix} -3x^2 + z^2 & 2zy & 2zx + y^2 \end{bmatrix}_p$ for $p \in f^{-1}(0)$.

$df_p = 0 \Rightarrow 2zy = 0 \Rightarrow y = 0$ or $z = 0$.

$$\begin{cases} z = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \\ \quad \text{and} \\ z = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0. \end{cases}$$ But $[0,0,0] \notin \mathbb{R}P^2 \Rightarrow \Leftarrow$.

$y = 0 \Rightarrow 2zx = 0 \Rightarrow x$ or $z = 0 \Rightarrow z^2$ or $-3x^2 = 0$ respectively $\Rightarrow z$ or $x = 0$ respectively. But again $[0,0,0] \notin \mathbb{R}P^2$.

Hence $df_p \neq 0 \ \forall p \in f^{-1}(0)$ and therefore $\overline{\varphi(C)}$ is a diff. submfd. of $\mathbb{R}P^2$.

Fall 2007 $\boxed{\#7}$

M, N compact, connected, n-dim'l mfds

$f: M \to N$ continuous. $H_n(f): H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$ nonzero.

$f_*: \pi_1(M; x_0) \to \pi_1(N, f(x_0))$. Claim: $f_*(\pi_1(M; x_0))$ has finite index in $\pi_1(N; f(x_0))$. Hint: covering of N.

Hatcher Thm 3.26 $\Rightarrow$ $H_n(M;\mathbb{Z}), H_n(N;\mathbb{Z}) = \mathbb{Z}$ if M, N orientable or 0 if nonorientable. Since $H_n(f) \neq 0$, it must be that M, N orientable. Thus we can let $K = \deg(f) := \deg(H_n(f): \mathbb{Z} \to \mathbb{Z})$

By classification of covering spaces, $\exists$ covering $p: \tilde{N} \to N$ s.t. $p_*(\pi_1(\tilde{N})) = f_*(\pi_1(M)) \leq \pi_1(N)$.

Thus, we want to show that $p_*(\pi_1(\tilde{N}))$ has finite index, i.e, that $p: \tilde{N} \to N$ is a finite-sheeted covering.

Assume not, then $\tilde{N}$ not compact $\Rightarrow H_n(\tilde{N}) = 0$.

But then $H_n(f) = H_n(p \circ \tilde{f}) = H_n(p) \circ H_n(\tilde{f}): H_n(M) \to H_n(\tilde{N}) \to H_n(N)$ must be the zero-map, a contradiction.

?

M compact $\Rightarrow$ $\tilde{f}: M \to \tilde{N}$ not surjective.

Then let $q \in \tilde{N} - \tilde{f}(M)$. $\exists$ nbd U of q that doesn't intersect $\tilde{f}(M)$. Let ω be supported on U.

Then $\int_M \tilde{f}^* \omega = 0 \Rightarrow \deg(\tilde{f}) = 0$.